Magnetic Modelling

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Abstract: The magnetic surface field of a star is constructed by a straight-forward calculation underlying a model of spatially distributed magnetic charges with their superposed potentials, for which the theoretic foundation is given.
The calculation is realized by a computer program, which fits the calculated phase curves to the observed ones by variation of parameters and iterative least squares optimization.
The magnetic map of the star is drawn on the base of the model by a few parameters. From the map the phase curves and the profiles of the spectral lines are derived.

1. Introduction

The ***structure of the magnetic field*** distribution of a star over its surface is hidden from observation by a lot of information destroying processes, leaving only the time-dependent modulation of the integral radiation flux by the star's rotation open for the disentangling of topographical features like the magnetic poles, because there is no direct way to draw a map of the star. For the reconstruction of the original surface distribution from the final observational values all these processes have to be inverted. The inversion, however, has no definite solution and is more or less impracticable. The difficulties bound to the generally ill-posed inverse problem are well-known and have been considered by *Khokhlowa* et al. [1] or by the authors in [2,3].
In contrary to this a straight-forward calculation can be carried out in any case. Assuming physically reasonable conditions we use only a few parameters to construct a magnetic map at the surface of the star. This is a matter of trial and error. At first a ***model*** will be constructed using reasonable but, nevertheless, arbitrary parameters. In this field *Bagnulo* et al. [4,5,6] have developed independently of ours an effective method, that has the same aim but other conditions.
By ***variation*** of the parameters and iteration a fitting of the calculated phase curves to the observational data can be achieved using the least squares optimization.

2. The calculation of stellar magnetic fields

The calculation of magnetic fields in stars has an old history. We refer here especially to the papers of *Oetken* [7,8], who modeled the star as an equatorially symmetric rotator for different cases (53 Cam, α^2CVn, βCrB, HD 71866, HD 32633, 73 Dra, 49 Cam, HD 126515, HD 98088, HD 49976, HD 24712, HD188041, HD 111133, HD 153882, HD 125248), which gave the impetus to the present paper. *Oetken* relates to *Krause, Rädler* et al. [9,10], who attempted to calculate the magnetic field structure of a star as generated by the action of a dynamo.
The solution of the ***hydromagnetic differential equations*** of the dynamo is displayed as a row of *Legendre functions*, the coefficients of which have to be determined like free parameters using numerical fitting procedures. The set of constants renders an analytical representation of the distribution of the field over the sphere but for its own does not allow any insight into the physical background. Besides of this, the field is specialized to the physical conditions of the dynamo and does not account for otherwise generated fields.

It should be mentioned that the calculation of stellar surface fields is valid for all hydrodynamic processes, which may cause ***magnetic fields as well as velocity fields***. Both fields – as well magnetic or/and velocity ones – produce shifts of the line profiles in the spectrum.

The ***magnetic field***, of course, originates in the interior of the star and penetrates the spherical surface of the star's atmosphere. Only from this location the magnetic field can be observed by the Zeeman displacement of spectral lines. The mapping of the surface field is the two-dimensional cartographic arrangement of the magnetic features of the outermost layer of the star. Any conclusion from the outer field to the inner one lacks of information if therefore no physical grounds is given. In any case we have to take the whole spatial distribution of the field into account. The magnetic field itself is a ***vector field***, which is defined completely by its *sources* and *whirls*. If we know the magnitudes and the spatial locations of them we can calculate the components of the field vector in any point of the surrounding space.

Sources and whirls constitute different kinds of field generators. Sources exist as individual monopoles, from which the field lines diverge radially, whereas whirls are circulated by closed field rings with a left or right handed rotation around an axial vector. The interaction of both sources and whirls is governed by Maxwell's equations and Ohm's law [Rädler 10], which gives the condition for the excitation of a dynamo in the electro-conducting turbulent medium of the star. This is well established for the sun.
The magnetic stars, which we investigate, are mainly A-stars. They show a very quiet behavior without turbulence, so that a recent dynamo cannot act. Obviously, we have to deal with a ***long-living permanent magnetism***. Therefore, we restrict the following considerations of the magnetic field structures in stars only to the stationary state, which is relevant for the mapping. In stationary conditions sources or whirls can exist separately. Sometimes only one kind of them is present.

Knowing that magnetic sources do not occur in reality, the stationary field is represented only by the whirls. With $\boldsymbol{B}$ as the field vector the absence of sources is expressed by the relation

$$\mathrm{div}\,\boldsymbol{B} = 0 \quad . \tag{1}$$

This means, that the magnetic field lines are circularly closed and have no beginning and no ending point. However, we can set such points artificially, if we cut up the ring of a closed field line, whereby a magnetic dipole is created. Thus a full analogy to an electrical field can be brought about.
Such a magnetic dipole is self-consisting like an electric dipole of two oppositely signed charges. The difference to the formal agreement of electrical and magnetic fields is that electrical charges are real individual sources but the analog magnetic charges Q are only virtual sources, which are coupled steadfastly to a dipole. However, the magnetic dipole is a real physical quantity with a *magnetic moment*

$$\boldsymbol{M} = \mathrm{Q}\,\boldsymbol{l}\ , \tag{2}$$

where Q is the "magnetic charge" and $\boldsymbol{l}$ is the length difference of the dipole center to one of the two charge locations. Thus the magnetic moment is a vector and undergoes all rules of vector algebra. This has the following consequences:

1. The magnetic moment produces a magnetic field environment of dipole structure.
2. The spatial vector fields of the dipoles superpose by vector addition.
3. The sum of several magnetic moments at the same location yields a resultant magnetic moment maintaining its environmental dipole structure.
4. The length $2\boldsymbol{l}$ spanning the distance between the virtual magnetic charges is an infinitesimal quantity $\boldsymbol{l} \to \mathbf{0}$ for the mathematical dipole, but can take on real significance for separated charges as realized in form of a rod magnet.
5. The virtual magnetic charges of dipoles and multipoles may formally be treated like separated individual field sources with arbitrary spatial distribution – provided the coupling of pairs with opposite sign and the sum of all charges being zero according to (1) is preserved:

$$\sum_i \mathrm{Q}_i = 0 \tag{3}$$

3. The physical significance of magnetic charges

The treatment of the *magnetic charges* as individual and separated field sources renders an important advantage because the arrangement of the locations in the star's body becomes very simple: each location of a charge is determined by the 3 spatial coordinates. The magnetic moment of the dipole as a vector is defined by 2 point locations or 6 coordinate values. There is no restriction to a mathematical dipole or to any spherical or axial symmetry. Dipoles and multipoles might be decentered anyhow. Even sunspots as narrowly located sources under the sun's surface may be described easily.

Equation (2) is derived for a magnetic dipole. However, it might be understood also as the magnetic moment $\boldsymbol{M}_i$ of a single charge Q_i in the distance $\boldsymbol{l}_i$ from the center of the sphere. By this way the advantage of the spatial arrangement of magnetic charges is preserved. Then the magnetic dipole moment $\boldsymbol{M}_d$ is the vector sum

$$\boldsymbol{M}_d = Q_1 \boldsymbol{l}_1 + Q_2 \boldsymbol{l}_2 \quad \text{with} \quad Q_2 = - Q_1 . \qquad (4)$$

But now we have to distinguish between poles and charges. The field strength at the poles, which we reduce from observation, is not a primary magnitude but only a derived one. The primary magnitude is the *magnetic moment* $\boldsymbol{M} = Q\,\boldsymbol{l}$, from which all other magnitudes of the magnetic field are derived. These magnitudes have often be confused, so that the physical dimensions of them should be born in mind.

By astronomers the *magnetic field strength* $\boldsymbol{B} = \mu\ \boldsymbol{H}$ ($\boldsymbol{B}$ magnetic induction, $\boldsymbol{H}$ magnetic field strength, μ magnetic permeability) is usually measured in the unit GAUSS. Not differing from this habit, the magnetic charge at the center of a sphere of radius R with the field strength $\boldsymbol{B}$ at the surface is

$$Q = 4\,\pi\,R^2\,|\boldsymbol{B}| . \qquad (5)$$

Then the physical dimension of the magnetic charge is *field strength* times *surface area* (or in units: GAUSS $*$ m^2). Likewise, the dimension of the magnetic moment is: *field strength* times *volume* (or: GAUSS $*$ m^3):

$$|\mathbf{M}| = 4/3\,\pi\,R^3\,|\boldsymbol{B}| \qquad (6)$$

The magnetic charge produces a central symmetric potential U at the surface of the sphere of the radius R:

$$U = Q/(4\pi\,R) \qquad (7)$$

If the charge is displaced from the center of the star, the polar coordinates (r radius, ϕ longitude, δ latitude) determine its point of location. Then by transformation to Cartesian coordinates

$$\begin{aligned} x &= r \cos\delta \cos\phi \\ y &= r \cos\delta \sin\phi \\ z &= r \cos\delta \end{aligned} \qquad (8)$$

we have with a = r/R as the fraction of the star's radius the distance from the center to the i-th point of the source

$$r^2 = R^2\,[\,(\cos\delta \cos\phi - a_i \cos\delta_i \cos\phi_i)^2 + (\cos\delta \sin\phi - a_i \cos\delta_i \sin\phi_i)^2 + (\sin\delta - a_i \sin\delta_i)\,] . \qquad (9)$$

The potential of the i-th charge is

$$U_i = Q_i/(4\pi\, r_i)\ . \qquad (10)$$

The potentials of several charges are superposed linearly:

$$U = \sum_i U_i \qquad (11)$$

Especially the potential U_d of a dipole with the charge Q and r_+, r_- for each source is given by

$$U_d = (Q/4\pi)\,(1/r_+ - 1/r_-)\ . \qquad (12)$$

4. The construction of a potential field of a magnetic charge

From this scalar potential the field strength is derived by the gradient

$$\boldsymbol{B} = -\ \mathrm{grad}\ U\ . \qquad (13)$$

The gradient is a vector of 3 components, which span a space with 3 orthogonal unity vectors as Cartesian or spherical coordinates.

In Cartesian coordinates x,y,z we have with the unity vectors ***i,j,k***

$$\mathrm{grad}\ U = \frac{\partial U}{\partial x}\boldsymbol{i} + \frac{\partial U}{\partial y}\boldsymbol{j} + \frac{\partial U}{\partial z}\boldsymbol{k}\ . \qquad (14)$$

Likewise, we have for each point of the sphere in the polar orthogonal system of radius r, longitude ϕ and latitude δ the gradient

$$\mathrm{grad}\ U = \frac{\partial U\, dr}{\partial r\, dx}\boldsymbol{i} + \frac{\partial U\, dr}{\partial r\, dy}\boldsymbol{j} + \frac{\partial U\, dr}{\partial r\, dz}\boldsymbol{k} \qquad (15)$$

If we use for polar coordinates only the radius r and simplify the constant with the charge to $C = -\ Q/4\pi$, then the potential $U = -\ C/r$ yields (16)

$$\frac{dU}{d\,r} = \frac{C}{r^2} \qquad (17)$$

The differential quotients, that gave the gradient along the 3 orthogonal polar coordinates, are:

$$\begin{aligned} \boldsymbol{B}_r &= \partial U/\partial r = (C/r^3)\,[\cos\delta\,(\cos\phi + \sin\phi) + \sin\delta] \\ \boldsymbol{B}_\varphi &= \partial U/\partial\phi = (aC/r^3)\cos\delta\,(\cos\phi - \sin\phi) \\ \boldsymbol{B}_\delta &= \partial U/\partial\delta = (aC/r^3)\,[\cos\delta - \sin\delta\,(\sin\phi + \cos\phi)] \end{aligned} \qquad (18)$$

These equations are the basic relations for the calculation of the magnetic field strength distribution over the star's surface for a single monopole. The differential quotients represent the 3 coordinates of the magnetic field at the surface of the star, which constitute the field vector. The mapping of the magnetic surface structure relates to these values.
In the case of a dipole a superposition of two monopole fields of opposite sign takes place. The summation of the fields of monopoles can arbitrarily be continued. Only for magnetic fields the pair-like combination of magnetic charges has to be obeyed, because the sum of the charges must be zero.

For the practical calculation on a computer the surface has to be divided in n*2n surface elements, which are arranged as a quadratic matrix with the rank 2n. Using the normal coordinates the longitude is divided in 2n and latitude done in n parts.
The 3 components of the magnetic vector are stored and may be recalled for other computations. By combination of the vector components the absolute value B_s is derived

$$B_S = \sqrt{(\boldsymbol{B}_r^2 + \boldsymbol{B}_\phi^2 + \boldsymbol{B}_\delta^2)} \ , \tag{19}$$

which is regarded as the surface field in one of the elements.

5. Observation of the integral radiation from the star

The calculation of the magnetic field strength renders a triple of values to every point of the stellar surface. However, the visibility of such a surface point depends on a lot of conditions bound to geometry, phase and physics of the star. In numerical computation such point is the center of an element.

The resultant values of the magnetic field strengths distributed over the stellar surface represents the map of the star $B(\phi,\delta)$. The globe of the star is seen by the observer under different aspects, caused by its rotation and the inclination i to the rotational axis. Besides of this, the visible disk is vignetted by the limb darkening according to the empirical formula with ε denoting the angle from the center of the disk

$$k = 1 - 0.4 (\cos \varepsilon) \ . \tag{20}$$

For the visibility of the star by the observer we define a window function W, which contains the inclination i, the projection of each surface element to the line of sight, and the limb darkening.

$$B_{e,s}(t) = \frac{\int_{\delta=-\pi/2}^{\pi/2} \int_{\phi=0}^{2\pi} B(\delta,\phi)\, W(i, \delta, \phi - t)\, d\phi\, d\delta}{\int_{\delta=-\pi/2}^{\pi/2} \int_{\phi=0}^{2\pi} W(i,\delta,\phi - t)\, d\phi\, d\delta} \tag{21}$$

This is the general relation between the magnetic field map and the observable *integral radiation flux* of any magnitude over the visible stellar surface, which we use especially for the magnetic field with all its vector components. This formula holds for the effective field B_e and for the surface field B_s after equation (19). The integral formula gives the integral mean of the disk seen by the observer and contains the convolution integral, which represents the rotation of the star with its map $B(\phi,\delta)$ behind the window $W(i, \delta, \phi)$. The denominator makes the normalization.

For the numerical calculation by a computer we replace the integral transformations by matrix multiplication. Therefore, the map will be discretized into surface areas as matrix elements, each element representing the integral mean value of this area.

6. The effective magnetic field $\boldsymbol{B}_e$ and the mean surface field $\boldsymbol{B}_s$

The magnetic field strength, which the observer measures by the Zeeman splitting of spectral lines, is the result of
1. projection by the coordinate orientation,

2. weighting by different areas of the elements and
3. vignetting by limb-darkening.

The resulting vector of the magnetic field by integration over the visible disk of the star is orientated anyhow, but only the projection to the line of sight to the observer gives the so called "longitudinal field" $\boldsymbol{B}_{e,}$ which is the mean value of the radiation from all visible elements influenced by the above mentioned conditions. The averaging rises some physical problems which we have to consider in the following.

Usually we measure the (effective) stellar magnetic field from the Zeeman displacement of the gravity centers of the line profiles of oppositely circularly polarized light. What we call the "effective magnetic field" is not a mean value but already the result of weighting and convolution of the radiation flux containing the magnetic field information about the form and the position of the profiles of all surface elements. In principle, the transmission of the flux through the atmosphere has to be treated correctly by the methods of radiation transfer theory, rendering the spectral dependence of the limb darkening.

In our computing program we relate to the fact, that the gravity center of two profiles of different height and position is given by the mean of the centers weighted by the profile integrals. Thus, we weight the magnetic field vector, projected onto the line of sight, of all surface elements with their spherical projection and limb darkening and integrate them over the visible hemisphere.

The projection of the magnetic field vector related to each point of the surface is carried out by a scalar multiplication of the magnetic field vector with the vector of the line of sight

$$\boldsymbol{s} \;=\; \cos\delta\cos\phi\,\boldsymbol{i} \;+\; \cos\delta\sin\phi\,\boldsymbol{j} \;+\; \sin\delta\,\boldsymbol{k}, \qquad (22)$$

which yields the scalar field components F related to the 3 polar coordinates of the surface elements (index p)

$$\begin{aligned} F_r &= B_r\,(\cos\delta_p\cos\phi_p\cos\delta \;+\; \sin\delta_p\sin\delta) \\ F_\phi &= B_\phi\,(\cos\delta_p\sin\phi_p\cos\delta) \\ F_\delta &= B_\delta\,(\sin\delta_p\cos\phi_p\cos\delta + \cos\delta_p\sin\delta) \end{aligned} \qquad (23)$$

This projection allows the calculation of the longitudinal magnetic field $\boldsymbol{B}_e$. The components of the vector undergo the averaging by the integral equation (21).

Likewise, we can calculate the projection onto the plane perpendicularly to the line of sight, if we draw the scalar products of the two orthogonal directions to the vector $\boldsymbol{s}$ in equation (22). Thus, the linear polarization (including all 4 *Stokes* vectors) is respected.

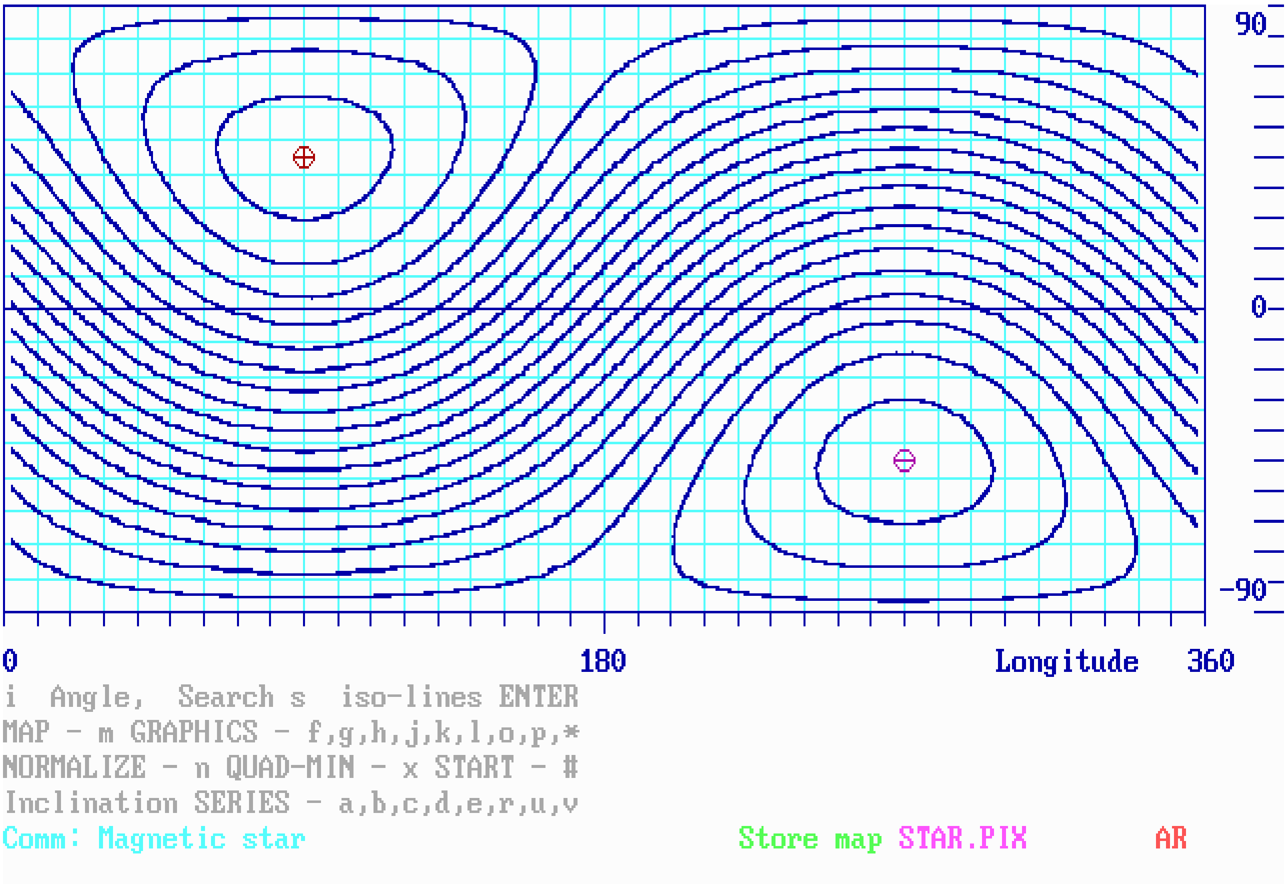
Here we apply the projection only onto the line of sight, because the observational material available to us has been obtained by means of a Zeeman-analyzer in the light path of a spectrograph, splitting the spectrum in two parts of left- and right handed circularly polarized light.

If we calculate the surface field B_s using the components (18) of the effective field components as in equation (19), then already the scalar intensity distribution of the absolute value of the vector is weighted, vignetted and integrated by equation (21).

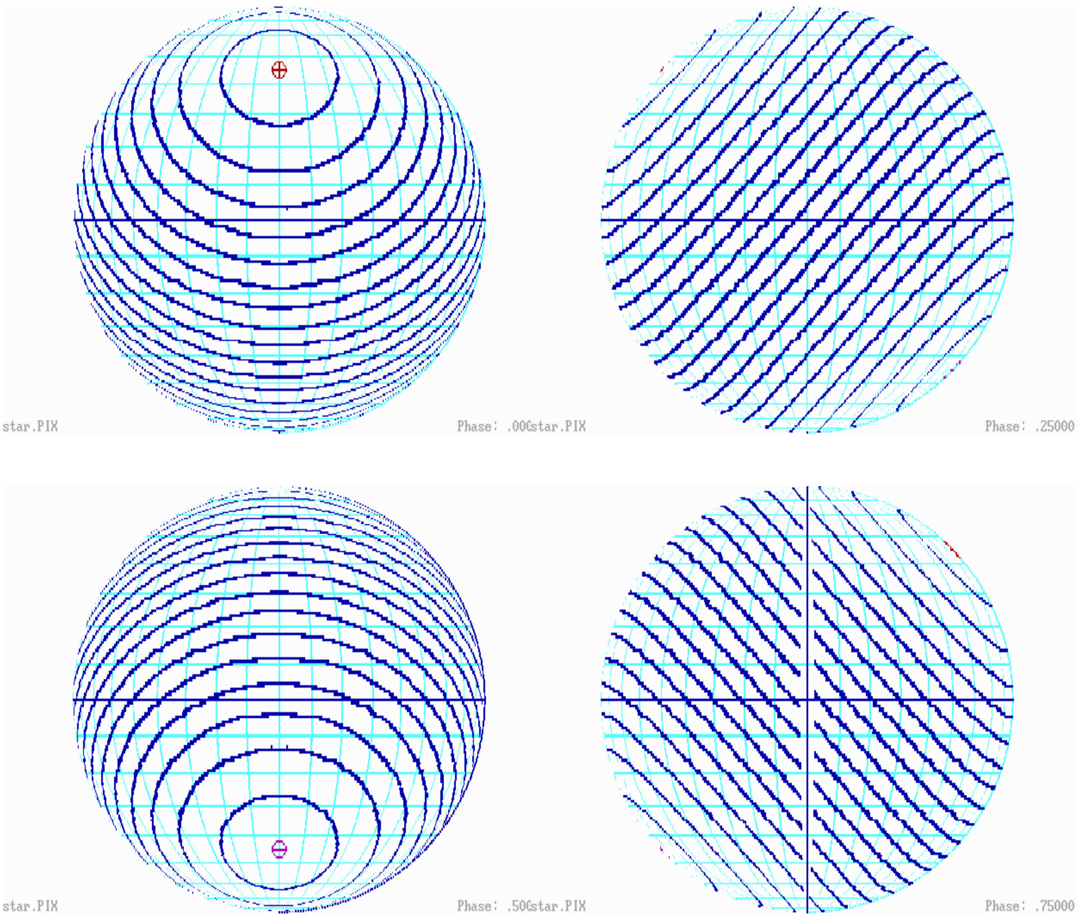
7. The distribution of the magnetic field over the stellar surface (magnetic mapping)

The calculated distribution of the magnetic field over the surface can be represented graphically. Thus a cartographic map of the star is drawn with topographical features of the magnitudes. Areas of the magnitudes are distinguished by colors and/or ***isolines***. The isolines are arranged as closed lines around the poles, which mark the most characteristic features of the map.

Mapping of a sphere is always a graphical problem. The plane (rightangular) projection corresponds well to the matrix arrangement of the surface elements. We demonstrate the mapping by a plane projection, which gives an overlook of the entire spherical surface of the star but has an extension of the longitude towards the poles. We show this at an arbitrary example:

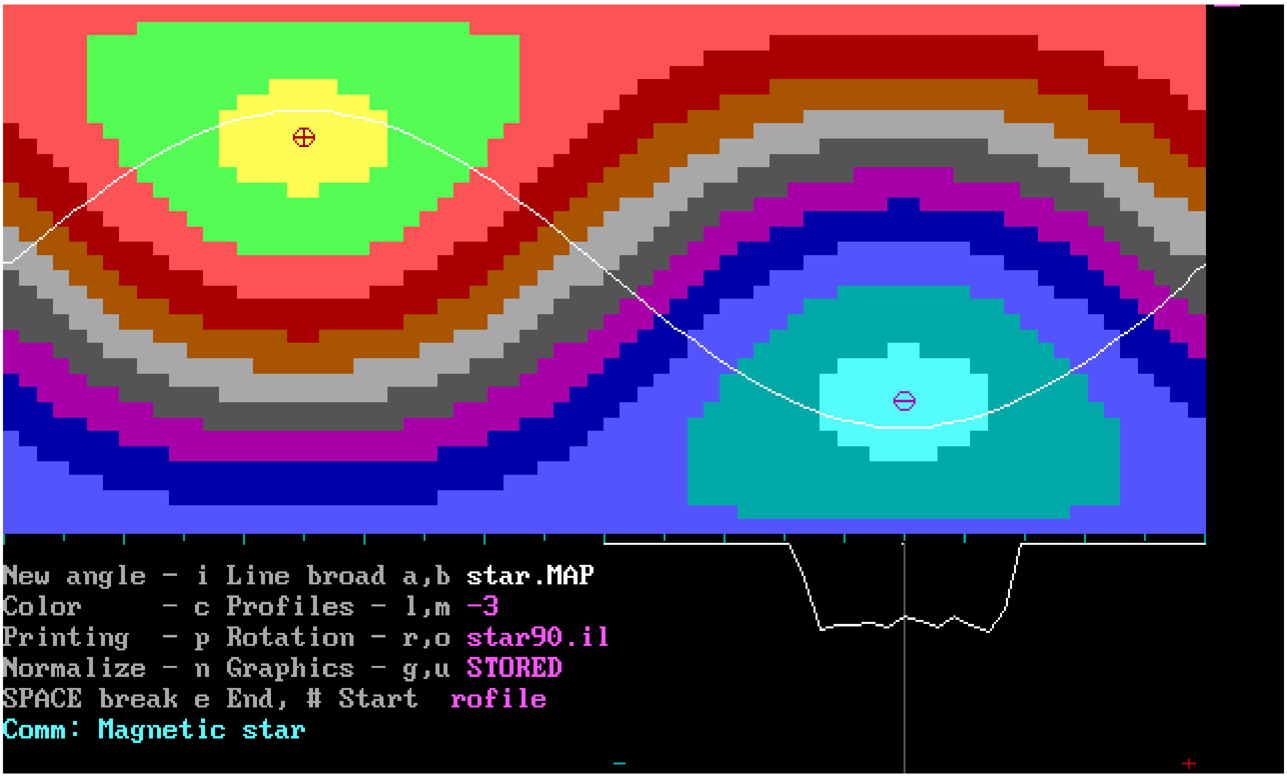


The sphere of the star is better shown in the correct perspective by transforming the coordinates into the orthographic equatorial projection. In this case only one half of the sphere can be seen so that the two opposite hemispheres of 180° longitude difference give all surface information.

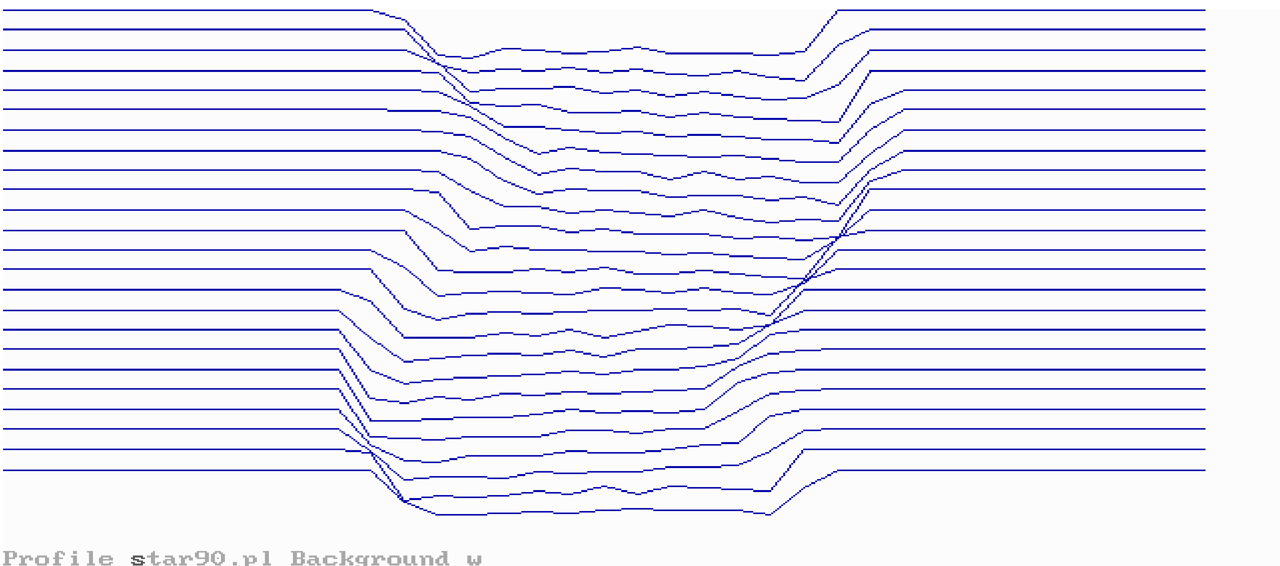


8. Phase curves and line profiles

The phase curves are calculated by a matrix convolution of the map resolved in matrix elements after the discretized integral transform equation (21). This corresponds to the rotation of the star, whereby the star shows periodically different aspects to the observer.
The radiation from the surface elements, which is weighted by geometry, projection and vignetting, can be evaluated in a statistical frequency distribution rendering the line profile.
For the above chosen example of a central dipole with the north pole at ϕ = 90° and δ = 45° the aspect to the equator with i = 90° gives a sinusoidal phase curve with the crossover between the positive and the negative poles at the phase 0.5. We see the characteristic rectangular profile with steep wings and plateau or dip within. The scatter is due to statistics and interference at the grating.



The group of profiles with the phase step 0.05 shows the influence of the aspect onto the profile:



The line profiles contain information about the aspect and the surface structure of the field.

The study of the profile shapes is very important for the correct assessment of the line position for the measurement of magnetic fields. Often asymmetric profiles occur. Then the gravity center and the extreme value may differ significantly. In special cases they are even in antiphase.

9. The complexity of magnetic and velocity fields

Once more we point to the fact that magnetic and velocity fields act in the same manner, for they shift the profile in the spectrum by the Zeeman displacement or by the Doppler effect. Normally these effects are combined in a close complex. The magnetic profile is convoluted by the rotation profile with all disturbances on account of surface inhomogeneities. In the case of an inhomogeneous distribution of chemical elements over the star's surface the radiation has a spotty character, which results in a spectral-photometric modulation by rotation. The photometric distribution acts like a transparency map covering the radiating surface. In the program a forth vector component is foreseen for the transparency, which allows photometric modelling for its own but also weighting of the magnetically relevant radiation by multiplication.
First tests have shown, that a complicate field structure may be attributed only to the chemical inhomogeneities, so that higher multipoles need not to be accounted for.

Thus magnetic mapping and Doppler mapping are closely connected and use the same formalisms and algorithms for the calculation.

10. Conclusions

The mapping of a magnetic star can be carried out on the basis of the mathematical treatment of a simplified model of a star with very few parameters in a straight-forward calculation. The model has to be brought in agreement with the real observations at magnetic stars. For this purpose a suited program is used as a tool. Magnetic and transparency inhomogeneities in the star are not only formally but also physically connected. The pole region is usually also a region of accretion or depletion of elements, from which the radiation with the spectral information about magnetic field and velocity goes out.
The observational material about chemically peculiar stars, which has been compiled since more than fife decades, needs devices and computer algorithms for its analysis in order to investigate the stellar magnetism, its origin, its evolution, and its boundary conditions.

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