A Physically Founded Analytical Representation of the Characteristic Curve [1]

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ABSTRACT. A simple few-parameter formula of the characteristic curve is derived on the basis of the theory of the photographic process, making use of special approximations that allow for elementary integration.
The formula is suited for interpolation of the characteristic curve and is appropriate for computerized sensitometric evaluation.

0   Introduction

The mathematical formulation of the characteristic curve in dependence on all influencing parameters is a main goal in the field of scientific photography; moreover, it may be of great importance, too, for the application of the photographic layer as a light detector in measuring techniques such as in astrophotography.
Especially, there is needed an almost simple – computer-appropriate – formula containing only a few parameters, which can be used to interpolate the sensitometric data of the empirically recorded characteristic curve and, conversely, to derive the exposure intensity values from the measured densities by the inverse function. Numerous proposals have been made for such formulae, as outlined in [8]; for example: the polynomial representation, the arctan-function, the Gauss error integral, the Baker function [1], and some more just phenomenological functions - all having actually no connection with the underlying physical process leading to the formation of the characteristic curve.

This paper is aimed at a   f o r m u l a t i o n   of the characteristic curve that reflects in principle the physical   f o r m a t i o n   taking advantage of some permissible simplifications to facilitate the mathematical treatment.

1   Physically founded statements for the derivation of the characteristic function

The photographic process, as well known, is an extraordinarily complex one, being in this field of knowledge the main object of research hitherto pursued, so that it seems to be quite impossible to put all these results together in a single formula. Such a global description of the photographic process will probably never be achieved at all.

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[1]ASTROPHOTOGRAPHY. Proceedings of the IAU Workshop, Jena, GDR, April 21-24, 1987, p. 83-90. Ed. S. Marx, Springer-Verlag Berlin Heidelberg New York London Paris Tokyo.
Scanned from the original publication by the author in 2010.
Revised and slightly accomplished by the author in 2018.

Nevertheless, simplified models can be constructed, showing the main features in a general way, and thus allowing a better insight into the process, which is otherwise obscured by its own complexity.

Therefore, we group the considerations that influence the formation of the characteristic curve in their order of importance:

1. The micro-processes in the crystal lattice of the silver halide during the exposure
2. The grain size distribution
3. The transfer of light radiation through the emulsion layer during the exposure
4. The shape of the developed silver grains
5. The development and further treatments of the photolayer
6. The after-treatment of the developed layer

In the following the points 5 and 6 are not considered closer because they are not essential for producing the typical form of the characteristic curve.

2 The microprocesses in the silver halide crystal

Since microprocesses in the crystal lattice are not directly observable, a broad latitude is left for hypotheses and theories. Recent ideas on the photographic microprocesses are based on the theories of GURNEY and MOTT [9] with the completion by MITCHELL [11] as summarized in [2]. In the present paper this wide field is not outlined. It should only be mentioned that in the past there turned out two main orientations of the theory:
1. the reaction-kinetics theory,
2. the statistical theory.

The first to attempt to describe the characteristic curve by means of reaction kinetics was SCHAUM [13] in 1909. The start of the statistical interpretation was made by SILBERSTEIN [14] in 1922. There is a lot of followers as outlined in [6]. Still recently, an example of a statistical description was given by FRIESER and KLEIN [17].
Both branches of the theory describe the same object and are at all (like the two sides of the same coin) equivalent. However, there are differences in describing the processing character of the creation of development centers, which, naturally, progresses in time. Thus, the statistical theory reflects the momentary situation of the process (like a single picture in a movie) and regards only the statistics of photons and grains, but does not account for the time. Therefore, this theory does not give any explanation of SCHWARZSCHILD's law, which in the kinetic theory already emerges plausibly out of the basic statements [3], but can be derived exactly [5, 7].

3 *Schwarzschild*'s law of blackening

As shown in [3] the product of power functions Et^p (E is intensity, t time, p SCHWARZSCHILD exponent), introduced by SCHWARZSCHILD as an independent variable of the density function, can be interpreted as the result of a multi-step kinetic reaction of the build-up of latent image specks, which trigger the reduction of the whole silver halide grain to metallic silver as development centers.

The explanation is given plausibly: The electron concentration c produced by the photoeffect in the lattice of the silver halide acts on each step of the reaction chain by discharging the center, resulting in a transition from one step to the higher one. Regarding the electron concentration c and the acting time t, the total probability for the transition of n steps producing a constant effect is the power product

$$c^n t^n = \text{const.} \tag{1}$$

If on one step – obviously the first – equilibrium is attained, then the time order reduces by one step, yielding

$$c^n t^{n-1} = \text{const.} \tag{2}$$

In exposure ranges whith an electron concentration c proportional to the light intensity E (long-time exposure, relevant in astrophotography), SCHWARZSCHILD's law

$$E t^p = \text{const} \tag{3}$$

follows with

$$p = 1 - \frac{1}{n}. \tag{4}$$

In the case of 4 steps we have the SCHWARZSCHILD exponent $p = 0.75$.

The quantity n is the reaction order of the centers representing the difference between the mean reaction order of the centers contained on all grains of the emulsion and the order $n = 4$ at which developability is reached.

The photographic effect of the exposure depends on the SCHWARZSCHILD product

$$H(E, t) = E t^p \tag{5}$$

as an independent variable called the *effective light quantity*.[2]

The average number $\overline{z}$ of centers per grain depends on the effective exposure H linearly with a factor of proportionality[3] ε raised to the power of n,

$$\overline{z} = \varepsilon H^n. \tag{6}$$

4 The dependence of the sensitivity on the grain volume

The photoeffect affects the whole volume of the silver halide grain supplying the electrons needed for the build-up of development centers. So it is plausible but also experimentally proven that the sensitivity of a grain (in a wide range of small and average-sized grains) increases proportionally with the volume,

$$\varepsilon^n = \eta V, \tag{7}$$

where η denotes a coefficient for the sensitivity density and V the volume. Thus, the number of development centers can be written

$$z = \eta V H^n. \tag{8}$$

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[2]Remark of the author in 2018:
$H(E, t)$ may be replaced by a more general expression: $H = [(1 + \varepsilon E)^b - 1]\, t^p$ with $1/2 \leq b \leq 1$.
References: [3], [5]. INTERNET: `ewald-gerth.de/22abs.htm`
[3]The exponent ε is in $H(\varepsilon E, t)$ implicitly bound to the intensity E – but also used explicitly (6).

5 The relation between the area density of the grains and the density

The developed grains of a photographic layer are distributed statistically with respect to their sizes and positions. The POISSON probability $P(z)$ of the occupation z of one cell of the honeycomb-like imagined layer by grains is[4]

$$P(z) = \frac{\overline{z}^z \mathrm{e}^{-\overline{z}}}{z!} = (\overline{z}^z \exp(-\overline{z}))/z!, \tag{9}$$

with $\overline{z}$ denoting the mean occupation.

Since already o n e grain in the cell stops the light penetrating through the layer we have with $z = 0$ the transmission probability

$$P(0) = \exp(-\overline{z}), \tag{10}$$

called the *transparency.*

By definition of the density D as the decadic logarithm of the transparency there follows

$$D = \log(1/\exp(-\overline{z})) = \overline{z} \log \mathrm{e}. \tag{11}$$

Taking into account that the mean occupation $\overline{z}$ is given by multiplying the area density ϱ of grains by the projected mean square area $\overline{a}$ of one grain, we obtain the well-known formula of NUTTING [12]

$$D = \varrho \overline{a} \log \mathrm{e} \tag{12}$$

expressing a linear relation between density and area density of grains.

6 The relation between the occupation number of centers and the grain area density

The POISSON probability that a silver halide grain is capable of being developed can be written as

$$P_d = 1 - P(0) = 1 - \exp(-\overline{z}), \tag{13}$$

because at least o n e development center must be contained on the grain surface attainable to the developer.

Using the proportionality found by NUTTING we have with the equations (8) and (13), introducing further the saturation density D_∞, a first approach to the blackening function, yielding already the typical sigmoid curvature of the characteristic curve,

$$D(H) = D_\infty(1 - \exp(-\eta V H^n)), \tag{14}$$

which corresponds to the density formula of SVEDBERG [15].

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[4]Remark of the author in 2010: Alike the original print of the *Proceedings of the IAU Workshop* the formulae are written in one line, which is adequate to the usage in a computer program.

7 The influence of the grain-size distribution

The silver halide crystals grow differently in the process of manufacturing the emulsion. The crystallization can be comprehended as a kinetic process of forward and backward reactions, whereby the bigger crystals grow at the expense of the smaller ones. From experimental results it is known that the grain sizes have a scatter like a logarithmic normal distribution. It might be interesting, that an exponential function with negative exponent $e^{-\alpha t}$ shows in logarithmic coordinates a bell-shaped curvature.

Such a distribution can be approximated also by a sum of a few terms of exponential functions, which corresponds to the type of solution of reaction kinetics equations.[5]

In normal coordinates a bell-curve may be achieved approximately by a distribution function $\omega(V)$ represented in the form of the difference of two exponential terms [8],

$$\omega(V) = \exp(-\alpha_1 V) - \exp(-\alpha_2 V). \tag{15}$$

However, it can be shown that the second term in equation (15) may be neglected, so that the the one-term exponential function

$$\omega(V) = \exp(-\alpha V) \tag{16}$$

almost suffices.

The neglected term does not contribute considerably to the total density, because in the corresponding part of the distribution

1. the sensitivity of the grains is low,
2. the volume of the grains is small,
3. the share in the total number of grains is small.

The form of the characteristic curve is determined essentially by the large grains. Only in the region of the shoulder do the small grains become considerable.

The influence of the grain-size distribution may be formulated by the integral

$$D(H) = D_\infty(1 - N \int_0^\infty V^\kappa \omega(V) \exp(-\eta V H^n) \,\mathrm{d}V) \tag{17}$$

representing the summation of the contribution to the density distribution. In (17) N is a normalizing factor, and κ denotes the "form exponent" depending on the shape of the developed silver grains. In the case of a spherical volume the density contribution of a grain is proportional to the projection area, i.e.: $V^{2/3}$.

Such grains are produced by half-physical development condensing the entire silver of the grain at the development center. Carrying out physical development, all silver is precipitated out of the solution phase regardless of the different grain sizes, making the density contribution independent of the grain size with a form exponent $\kappa = 0$.

Chemical development produces sponge-like silver conglomerates whose light absorption is proportional to the quantity of silver contained in a grain, which, in the extreme case, makes the form exponent $\kappa = 1$. In the normal case one can expect a form exponent lying between $2/3 \leq \kappa \leq 1$.

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[5]Remark of the author in 2010: The mirror-symmetrical form of the logarithmic normal distribution comes out approximately for $\alpha_2 = 4\pi\alpha_1$. The factor 4π relates to the surface of the sphere by $4\pi r^2$.

The integration of (17) may be performed with any grain size distribution one would like. An interesting approach to the solution is the representation by a convolution integral.

A distribution function consisting of a sum of exponential terms can be integrated particularly easily in an elementary way.

The only example we state here is the one-term exponential function according to (16), which yields

$$D = D_\infty(1 - 1/((1 + \frac{\eta}{\alpha}H^n)^{\kappa+1})) \tag{18}$$

In the case of more summands in the distribution function we obtain as the solution a linear combination of terms like (18).

The ratio of the sensitivity density η and the grain-size distribution coefficient α proves to be a *sensitivity coefficient*

$$\varepsilon = \eta/\alpha, \tag{19}$$

which points to the strict correlation of sensitivity and grain size.

8   The influence of the radiative transfer

Passing through the opaque emulsion layer during exposure the light undergoes attenuation by absorption and scattering at the crystals of the silver halide grains. The correct attenuation function $\varphi(x)$ emerges as the solution of the integro-differential equation of the radiative transfer process.

A simple, but nevertheless sufficient solution is the exponential law of attenuation after LAMBERT

$$E_{\text{eff}} = E \exp(-\frac{\delta}{x_0}x), \tag{20}$$

x denoting the depth and x_0 the thickness of the layer, δ the optical density of the opaque emulsion coating, E the intensity and E_{eff} the effective intensity. The LAMBERT law holds better the more absorption predominates over scattering within the layer.

We restrict the attenuation function $\varphi(x)$ to the one-term exponential function because of the difficulties arising in analytical integration in the case of using more complicated functions.

The entire layer may be divided into an infinite set of infinitesimal layers, whose densities are superposed according to the elementary blackening function, yielding by summation of all partial densities the integral over the depth of the layer

$$D = D_\infty\Big(1 - N\int_0^{x_0} P(E\varphi(x),t)\,\mathrm{d}x\Big), \tag{21}$$

being P the POISSON-probability and N a normalizing factor.

In the case of equal grain sizes the evaluation of the integral (21) turns out to be an exponential integral function, i.e. a relatively complicated function [4]. Paradoxically, taking account of the grain size in the form of a sum of exponential functions leads to a simplification of the analytical formulation of the blackening function, which can be expressed in terms of just standard functions. (We are confronted with the same effect as known for integral transformations – the FOURIER and the LAPLACE transformations.)

9 A simple but physically founded analytical characteristic function

The integration over the grain-size distribution and the thickness of the layer can be together represented by the following double integral:

$$D = D_\infty\Big(1 - N\int\limits_0^{x_0}\int\limits_0^\infty V^\kappa\varphi(V)P(\frac{V}{\eta}E(x), t)\,\mathrm{d}V\,\mathrm{d}x\Big) \tag{22}$$

The integration of (22) with the simplifying restrictions
1. SCHWARZSCHILD's blackening law is valid,
2. the sensitivity is proportional to the grain volume,
3. the grain-size distribution follows a one-term exponential law,
4. radiative transfer is given by LAMBERT's law,
5. there is physical development with a form exponent $\kappa = 0$

yields the formula

$$D = \frac{D_\infty}{n\delta}\ln\frac{1 + (\varepsilon H)^n}{1 + (\varepsilon H)^n\exp(-n\delta)} \tag{23}$$

with $H = Et^p$ (the SCHWARZSCHILD product) as an independent variable.
Comparing with real photographic blackening curves this formula holds very well in the region of the toe and satisfactorily over the quasi-linear middle part of the curve, but deviates in the region of the shoulder. This is because of the simplifications listed above, which affect significantly only the shoulder of the curve. However, the torsional symmetry of the curvatures of the toe and the shoulder regions makes the mathematics of the formula simple and comprehensible.

Nevertheless, the integration could be performed with fewer simplifications. The integrability of (21) is preserved if we use for the grain-size distribution as many terms as one likes; but only the form exponents $\kappa = 0$ and $\kappa = 1$ allow analytical integration [8]. In every case a numerical integration could be performed, embracing all possible relations correctly. In all those cases the simplicity of a short analytical formula is lost.

10 The properties of the proposed four-parameter blackening formula

The formula (23) contains only 4 parameters:
1. the saturation density D_∞,
2. the opacity density δ,
3. the step order n,
4. the sensitivity ε.

Therefore, only 4 values of the density spread over the characteristic curve are needed in order to determinate the whole function, being sufficient for interpolation. With more values an optimizing calculation could be carried out.

The characteristic curve drawn from the formula (23) is quite symmetrical to the point of inflection. This becomes obvious if we take the slope by differentiation.
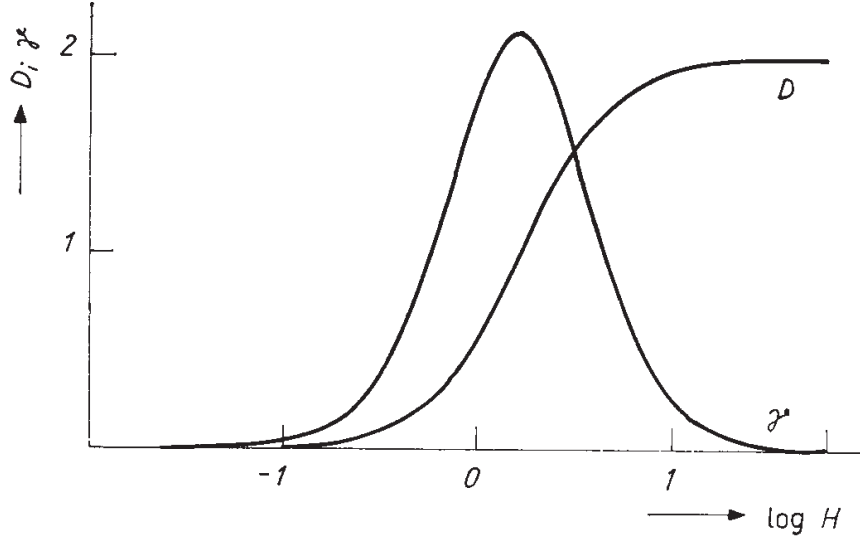
Fig 1   Characteristic curve calculated using equation (23) and the related slope after equation ((24), which exhibits a mirror-symmetrical shape. $D_\infty = 2$, $\delta = 1$, $n = 2$, $t = 1$.

The derivative

$$\gamma = \frac{D_\infty}{\delta}\Big(1/(1+(\varepsilon H)^n) - 1/(1+(\varepsilon H)^n \exp(-n\delta))\Big) \tag{24}$$

consists of a difference of two LORENTZ-like terms with a coordinate shift between them. The gradient at the inflection point is derived as

$$\gamma_{ip} = \frac{D_\infty}{n\delta} \tanh \frac{n\delta}{4}, \tag{25}$$

showing that the gradient can only increase up to a limiting value given by the ratio D_∞/δ.

The density at the inflection point is accurately

$$D_{ip} = \frac{D_\infty}{2} \tag{26}$$
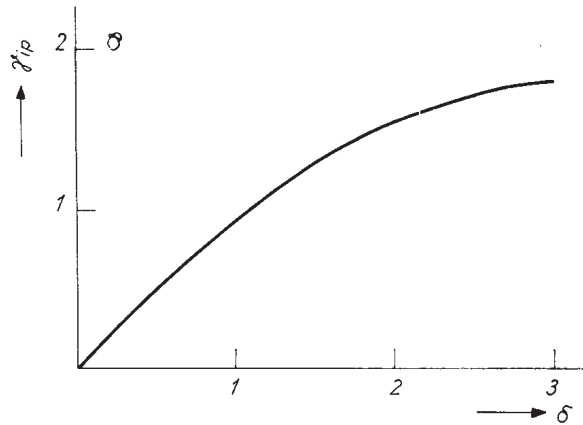
in accordance with the torsion symmetry of the curve.

Fig 2   Dependence of the gradient at the infection point of the characteristic curve on the opacity density of the emulsion layer. $D_\infty = 2$, $n = 2$.

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In order to demonstrate the influence of the parameters on the characteristic curve represented by (23) two families of curves are presented. Figure 3 shows the effect of the step order n.
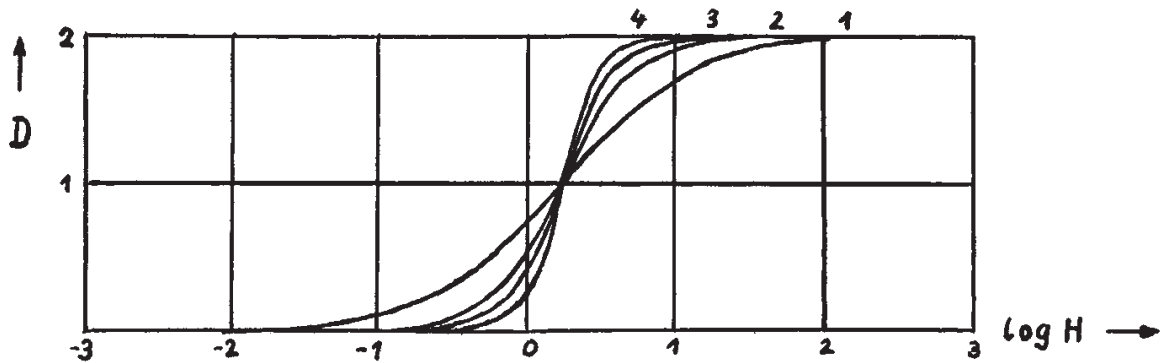

Fig 3   Family of characteristic curves by the parameter n.
    n = 1, 2, 3, 4. Further values: $D_\infty = 2$, $\delta = 1$, $\varepsilon = 1$

For reasons of the kinetic theory of the step-like build-up of developing centers [3,6] the step order is limited to $1 \leq n \leq 4$, n being near 1 for high-speed emulsions as used in astrophotography and being near 4 for unsensitized emulsions of high contrast as used for printing graphical purposes [10]. We can see that the contrast increases while the sensitivity decreases with growing values of n. This exponent has been used by de VAUCOULEURS [16] to represent the toe of the characteristic curve by a power function.

Figure 4 shows the effect of the opacity density δ. Because of the increasing absorption in the layer during the exposure the curve is stretched to a nearly linear middle part but also flattened with extension of the exposure latitude and, of course, shifted to the less sensitive side. This effect may be produced on a given photomaterial by dyes acting during the exposure. The dyes then can be washed out after the development, thus supplying a better transparency of the finally resulting photograph. Such a method is recommended for correct reproduction in the printing process.
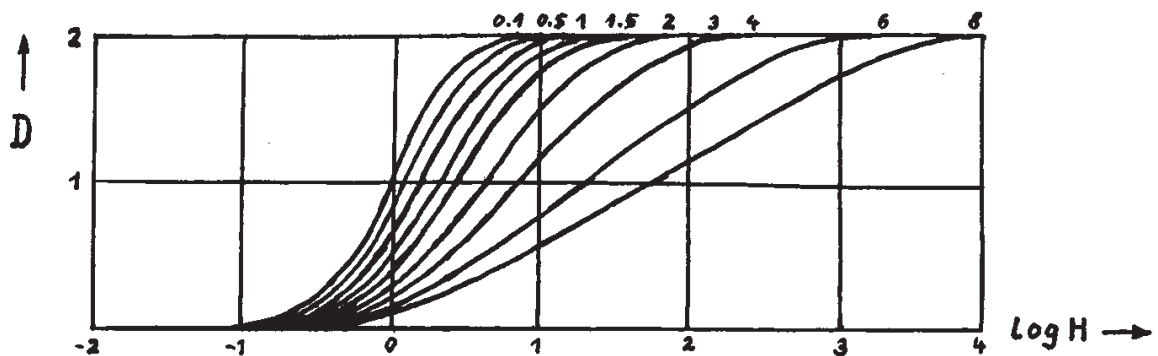

Fig 4   Family of characteristic curves by the parameter δ.
    $\delta = 0.1, 0.5, 1, 1.5, 2, 3, 4, 6, 8$. Further values: $D_\infty = 2$, $\varepsilon = 1$, $n = 2$.

The sensitivity parameter ε causes a shift along the $\log H$-axis; likewise the saturation density D_∞ stretches the ordinate.

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Both influences on the characteristic curve need not be demonstrated explicitly. It should, moreover, be considered whether the coefficient ε could be used as a measure of the sensitivity because it is physically founded and independent of particular practical references.

The formula (23) is appropriate for different analytical treatments requiring a mathematical formulation of the characteristic curve. Especially, the function (23) may easily be inverted, allowing for derivation of the exposure light quantity H from the density. The simplicity of the formula makes it well-suited to application in computer programs, especially when a subroutine of this formula is run frequently as, e.g., for the recalculation of the intensity from densitometer tracks.

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