Calculation of the integral magnetic field of a star accounting for the surface distribution of elements

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Abstract. The observable magnetic field of a star is the result of integration over its visible hemisphere, related to the information transferring medium: the spectral line profile. The hitherto practised simple integration of the magnetic field strength neglects the spotty face of the star and is physically wrong. Because of the topographically distributed line-generating elements in the stellar atmosphere, the contribution of all parts of the surface to the integration is different. For an effective computation, both the magnetic field and the element distribution are transformed from globes to Mercator maps and arranged as right-angled matrices. The numerical evaluation is performed by a special computer program, which uses matrices and vector algebra. The theory is based on the mathematical derivation of convolution integrals for the rotation of the star and the line profiles formed in its atmosphere, whereby the radiation from all surface areas in direction to the observer is integrated, accounting for the geometrical and radiation transfer conditions of the disk-like visible hemisphere and the element distribution of chemically peculiar (CP) stars. The computation starts from a given magnetic field structure on the surface of a star and progresses straightforward over convolution integrals to the phase curves of the integral magnetic field strength. In consideration of other approaches to the problem of field structure analysis, also the inversion of the convolution is discussed. [1]

Key words: CP stars, magnetic field, element distribution, line profile

1. Introduction

All information we get of a stellar object is comprised in its radiation. If we investigate surface phenomena of a star as brightness, temperature, but also magnetic fields and material movements, then the observation is restricted to the partially visible hemisphere with its geometrical, geographical, and physical conditions, which is mixed to one integrated stream of information with total loss of topographic details.

[1] Preprint. Submitted to: Magnetic Stars, Proc. Internat. Conf. SAO, Nizhnij Arkhyz, Russia, 28 August - 1 September 2006. Article available by www.ewald-gerth.de/119.pdf.

In the case of the magnetic field there exists no direct measurement from the Earth. The information transferring medium is the spectral line profile, which is produced in the star's atmosphere by atomic processes of special chemical elements distributed over the surface more or less inhomogenously. Thus, not the magnetic field itself is integrated, but the line profiles from all surface areas are averaged over the observable disk of the star.

In this paper the line-bound integration of the magnetic field over the visible hemisphere of the rotating star is described by mathematical treatment. Some account is given for the main algorithms used already in former programs (Gerth & Glagolevskij, 1997[2], 1998, 2000, 2001, 2004).

It is the aim now to make the algorithms understandable and reproducible.

2. Origin and theoretical construction of stellar magnetic fields

The origin of stellar magnetic fields is not clear yet and still under discussion. What we assume is that magnetic fields fill the universe since the Big Bang, which released mighty streams of electrically charged particles, all being on their way surrounded by circularly closed magnetic lines of force. The omnipresent magnetic fields propagate independently through space. They were captured during the condensation of intergalactic gas creating stars with frozen-in magnetic moments, which we call the *relict stellar magnetism.* On the other hand, a star could become magnetized even still recently by energy-driven motions of material in its plasma, so as it is realized in the Sun by a dynamo mechanism. A further possibility for the generation of a magnetic moment in a stellar body is the influence of an external field. This can be a wide-spread cosmic magnetic field ore a nearby magnetic companion.
Anyway, regardless of any origin of stellar magnetism, we are interested here only in the temporally stable magnetic field existing at the surface of the star with its topographic coordination, which we can observe by the Zeeman-effect as the magnetic detector in the stellar atmosphere.

Since the topographic details on the star are occulted for our observation by the physical integration of the total radiation, disentangling of the mixed information is nearly hopeless. The *inverse solution* of the integral equation is very problematic because of lack of information, but it yields as the result a topographic arrangement of the magnetic surface field in a cartographic map – though without any explanation of its origin.

Avoiding the problems of the inversion procedure, we take the cartographic map of the magnetic field structure on the star's surface as the given starting point for the derivation of the *integral magnetic field* in a so-called *straight-forward calculation*, which starts from a physically reasonable hypothesis and carries out the integration by a mathematical treatment on a theoretical basis.

The mathematical derivation proves the hypothesis by comparison with - and fitting to - the real observation, but it is suited also for controlling the inverse solution.

[2]The first poster calculated using our program was represented on the conference in Vienna October 27–29, 1997. (Available by www.ewald-gerth.de/91pos.pdf)

For the theoretical calculation of the magnetic field on the surface of a star there were proposed several different methods, which depend on the underlying physical approaches:

1. The magnetic field structure on the star's surface is a result of the solution of Maxwell's and Navier-Stokes' differential equations. This is realized e.g. by the Dynamo theory of Krause and Rädler (1980) with a clear physical origin of the magnetic phenomena. The solution is mathematically expressed in spherical harmonics for the magnetic vector field of the interior and the surrounding space including the surface of the star.

2. The magnetic field is calculated with hypothetically adopted magnetic poles on the surface of the star as parameters (Bagnulo et al., 1996), which will be varied and fitted to the real observation. The mathematical formulation uses spherical harmonics for the first orders as dipoles, quadrupoles, octupoles etc. and gives an analytical expression in form of expansions of Legendre's functions, but it lacks for an explanation of the physical origin of the magnetic field. Nevertheless, this approach renders a map of the magnetic field vector on the surface of the star.

3. The magnetic field is calculated on the basis of the potential theory, according to which every vector field is constituted by the field generation originating from sources and vortices. The mathematical treatment bases exclusively on vector algebra and does not need spherical harmonics. Therefore, the word *dipole* has another meaning than that of item 2: it is a combination of two magnetic monopoles with equal but opposite charges. The property of linear superposition of the fields allows combinations of numerous sources like dipoles to represent any magnetic field structures inside and outside the star including the surface as a vector field.
   This method to calculate the stellar magnetic field goes back to a proposal of Glagolevskij and Gerth, which they have described in several foregoing papers (Gerth & Glagolevskij, 1997, 2000, 2002, 2004,).

Here we list only the main methods of calculating magnetic fields, being discussed at the time. The magnetic map can be derived also by other methods. We start the further calculation of the integral magnetic field from that stage, where a *global cartographic map of the magnetic field structure already exists.* By this way the here proposed calculation of the integral magnetic field in its appearance by covering of the surface with chemical elements is valid generally.

Remark: In the following text we complete the descriptions of former papers essentially by outlining the base of the algorithms used in our computation programs. Repetitions are avoided as far as necessary for explanations at the corresponding places in the treatise. Thus, any description of our proposal for modeling magnetic field structures by the *Magnetic-Charge-Method* (MCD) is left out. Further, we restrict the analytical formulation of the integral magnetic field by convolution integrals to the so-called *effective magnetic field strength* $\boldsymbol{B}_{\text{eff}}$, which is the circularly polarized component of the Zeeman-displaced spectral line profiles (Stokes V) and may be generalized. Concerning the other Stokes-components I, U, Q, we refer to our former publications quoted above.

3. Geometrical conditions

The geometrical situation of the star in relation to the viewer has clearly to be defined, not only to avoid confusion but to get the best standpoint in a coordinate system suited for the mathematical treatment.

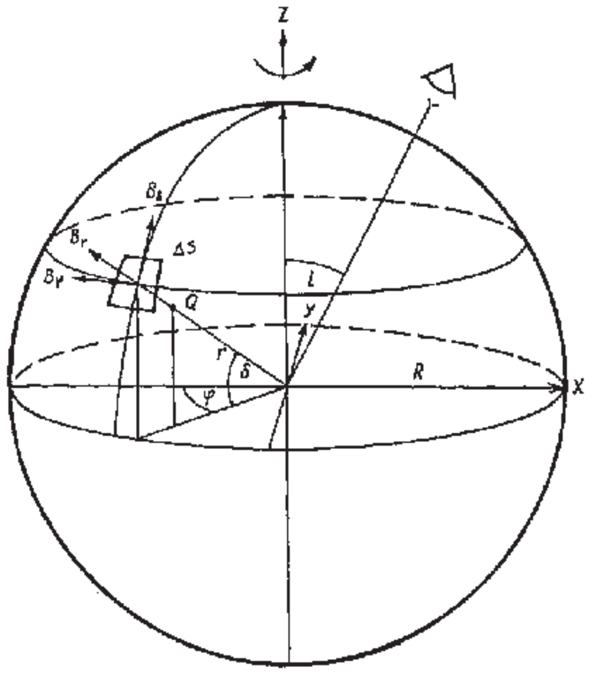

Figure 1. *The star is orientated in the Cartesian coordinate system with its rotation axis coinciding with the z-coordinate. The observer looks at the axis by the inclination angle i.*

As shown in Fig. 1, we use the following definitions:

1. The geometry of the star itself is given by the topographic structure on its globe with radius R, coordinated on the surface by its longitude φ and its latitude δ.

2. The orientation of the star to the viewer on the Earth is given by the inclination angle i, which is spanned between the line of sight to the star and the direction of its rotation axis.

3. The rotation axis coincides with the z-coordinate of the Cartesian system.

4. The momentary aspect of the star is given by its phase during the period of rotation. The phase angle determines the periodical progression in time by the characteristic difference (φ - t).

5. A point on the surface with the coordinates φ and δ is penetrated by a magnetic field vector with three components B_r, B_φ, B_δ.
   Q is the (virtual) magnetic charge of a source positioned in Cartesian (x, y, z) and spherical (r, φ, δ) coordinates (valid only for the MCD-method used by Gerth & Glagolevskij, 1997, 2000).

4. Coordination by a matrix scheme

The coordination between the globe and the Mercator map is essential to the mathematical treatment. The three Cartesian coordinates x, y, z are reduced to the two plane coordinates φ and δ, projecting the rotation axis of the globe to the latitude in the Mercator map. This gives the arrangement of the globe in Fig. 1 with a perpendicular rotation axis.

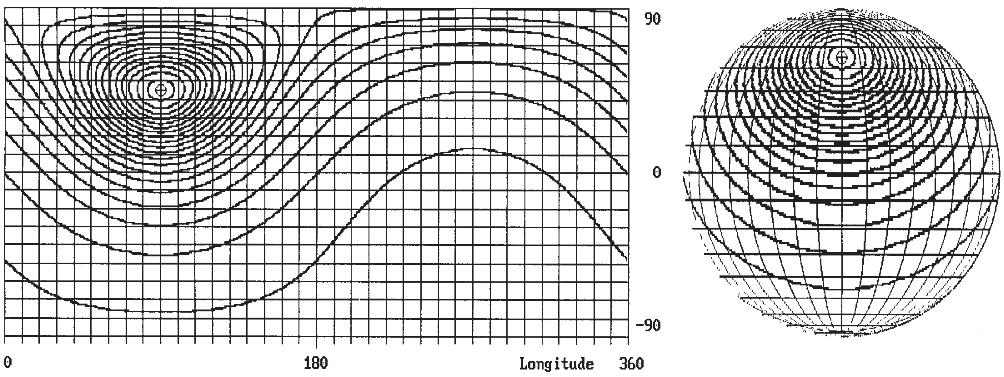

Figure 2. Map and globe of the field structure of a monopole.

As we see at Fig. 2, the spherical coordinate net of the globe is devolved to a rectangular one, constituting the scheme of a rectangular matrix

$$\boldsymbol{B}_{\delta\varphi} = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1\varphi} \\ B_{21} & B_{22} & \dots & B_{2\varphi} \\ \vdots & \vdots & \ddots & \vdots \\ B_{\delta 1} & B_{\delta 2} & \dots & B_{\delta\varphi} \end{pmatrix} . \tag{1}$$

The parameters of such a matrix element relate to the center of a spherical trapezoid area on the globe bordered by the coordinates φ and δ, which constitute the rows and the columns of the matrix of the magnetic field strength.

In order to get a clear coordination, we use for the indication of the matrix elements the symbols for the latitude $\mathrm{i} = 1 \cdots \delta$ and $\mathrm{k} = 1 \cdots \varphi$. The matrix elements B_{ik} are confined by the equidistant coordinate lines shaping squares, which correspond to spherical trapezoids with diminishing areas from equator to pole. They are orthogonal vectors of the magnetic surface field with 3 components:

B_r radial component
B_φ tangential component in direction of the longitude
B_δ tangential component in direction of the latitude

The three components of the magnetic vector belong to three cartographic maps and the corresponding matrices, the sum of which establishes a "vector-matrix".

The square area s of a cartographic surface element of the matrix depends on the radius of the star R, the latitude δ and the rank n by

$$s = \frac{R\pi^2}{2n} \cos\delta. \tag{2}$$

For the calculation of the area A we use in our program instead of the spherical trapezoid a plane one, whose tangential area is a bit larger than the square on the sphere. The difference between them diminishes with growing rank of the matrix and disappears in the infinitesimal case.

The trapezoid area of a surface element is also the concurrent factor, which determines the share of radiation from the globe contributed by one matrix element in equation (1). The variation of these areas from pole to pole is expressed by a row matric

$$(\boldsymbol{s})_\delta = (\ s_1 \quad s_2 \quad \dots \quad s_\delta\ ) \ . \tag{3}$$

This coordinate arrangement, however, is not compelling at all. Thus, Piskunov (2001) uses for the coordinated representation of the globe a slightly tilted rotation axis with partly view at one of the poles. The latitude zones there are divided by a number of equal areas in order to provide equal radiation from all surface elements.
A tilted rotation axis by the inclination angle i in order to put the line of sight in the z-axis of the coordinate system was used by Oetken (1977, 1979).

5. Direct integration of the magnetic field

Since all things we observe of a star from our distant position of the Earth are reduced to an one-dimensional information stream by averaging all surface details, it was obvious to represent this by integration of the observed magnitude over the disk of the star. Thus, some first attempts of modeling stellar magnetic fields were made by the same way.

Deutsch (1970) and later Oetken (1977) integrated the magnetic field over the visible hemisphere by the integral formula

$$\boldsymbol{B}_{\text{eff}}(t) = \frac{\int\limits_{\varphi=0}^{2\pi}\int\limits_{\delta=0}^{\pi/2} \boldsymbol{B}_z(t)\ g(t)\cos\vartheta\ \sin\vartheta\ \mathrm{d}\vartheta\ \mathrm{d}\varphi}{\int\limits_{\varphi=0}^{2\pi}\int\limits_{\delta=0}^{\pi/2} g(t)\cos\vartheta\ \sin\vartheta\ \mathrm{d}\vartheta\ \mathrm{d}\varphi} \ , \tag{4}$$

where z denotes the direction of the line of sight, and $g(t)$ is a weighting function describing the contribution of the surface elements to the "effective magnetic field strength" B_{eff}. The denominator makes the normalization. t marks the momentary phase, which determines B_z. The angles φ and ϑ are integration variables with respect to the line of sight - without any relation to the cartographic coordinates of the star's globe. The surface field is described by spherical harmonics. This makes the calculation of the phase curve rather difficult, because the rotation axis is tilted to the perpendicularly arranged line of sight, and the problem is solved by a coordinate transformation of spherical harmonics.

Objections could be raised also to the direct integration of the magnetic field over the star's surface, neglecting the vectorial character of the magnetic field and the information transfer through the spectral line profile. Therefore, Oetken (1977) calls this rightly a "rough way".

6. The observation window

The sight at the star acts like a window. We see only one hemisphere of the star with all geometrical conditions of perspective distortions and occultation, e.g., by limb darkening - or distribution of chemical elements.

Referring to Oetken (1977), who relates to Pyper (1969), then the weighting function g in equation (4) is separated in two factors

$$g = g^*(\vartheta) \cdot W_\lambda(t, \vartheta, \varphi), \tag{5}$$

the second of them expressing the wavelength-dependent radiation geometry and the first one the limb-darkening function in respect to the line of sight

$$g^*(\vartheta) = (1 - \mu + \mu \cos\vartheta)(1 - \kappa + \kappa \cos\vartheta) \tag{6}$$

with μ and κ as empirical coefficients.

6.1. The window function

For the visibility of the star by the observer, the authors (Gerth & Glagolevkij, 2000) define a window function $\mathrm{w}(i, \varepsilon, \delta, \varphi)$ containing the independent variables

i inclination	δ latitude
ε limb darkening	φ longitude .

The window function $\mathrm{w}(i, \varepsilon, \delta, \varphi)$, however, is very substantial, for it comprises not only the geometrical aspect of the visible hemisphere of the star but the whole physics of radiative transfer through the stellar atmosphere.

6.2. Limb darkening

Keeping in mind that nature is always richer than any attempt to describe it even only roughly, we restrict ourselves to a simple formula like equation (5), reducing the number of parameters and economizing with computer time.
Instead of equation (5), however, we use the limb-darkening formula

$$\varepsilon(\vartheta) = \frac{I(\vartheta)}{I(0)} = \frac{2}{5}\,(1 + \frac{3}{2}\,\cos\vartheta), \tag{7}$$

which is taken from Voigt (1980) and represents the truncated expansion of Eddington's approximation for the anisotropic part of the radiation transfer through a grey atmosphere, covering in a spherical layer the photosphere of a star. In this formula I is the intensity, and ϑ is the angle between the line of sight and the radius vector directed from the center of the globe to any point on the surface. Because of the circular symmetry, $\sin\vartheta$ is the relative distance from the center of the disk, where the intensity is $I(0)$.

It should be mentioned here that equation (5) can be exchanged by other solutions of the radiative transfer, accounting for atomic processes, isotropic scattering, and line formation in the atmosphere. This is a possibility to prove the transfer theory comparing it to observation at a convex atmosphere layer.

6.3. The geometry of the disk

The star seen as a disk is a projection onto the globe of the hemisphere in direction of the line of sight. This is like a cap of half a sphere the globe is sliding about. All surface elements of the globe with their coordinates φ (longitude) and δ (latitude) are projected to the line of sight, which penetrates the surface with the angle i in respect to the rotation axis by the latitude $\delta_i = 90^\circ - i$.

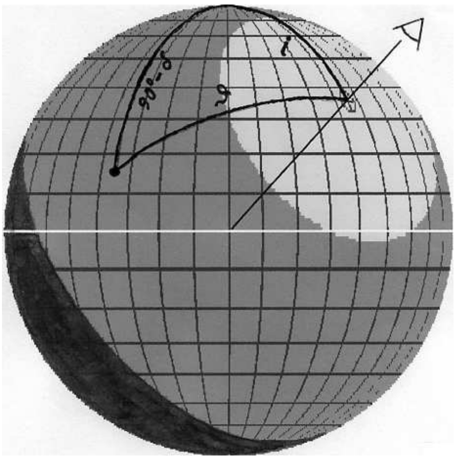

Figure 3. The star is coordinated like Fig. 1. The distance from the penetration point at the coordinates φ_i and $\delta_i = 90^\circ - i$ through the surface to any other point of the surface with the cartographic coordinates φ and δ is derived from the polar spherical triangle using the spherical cosine theorem.

The projection causes a tilting of the plane of any surface element diminishing its aspect area by a factor $\cos\vartheta$ - with ϑ as the spherical angle on the globe between the penetration point of the line of sight and the cartographical position of the surface element. Thus, the projection of the surface elements around the axis of the line of sight has an axial symmetry with circles of equal projection. The distance of the element to the line of sight is given by the spherical cosine theorem

$$\cos\vartheta = \sin\delta\cos i + \cos\delta\sin i\cos\varphi. \tag{8}$$

$\cos\vartheta$ has a positive value on the hemisphere towards the viewer, at the backward hemisphere it is negative, which indicates the neglecting of this part.

Combining equation (8) of the aspect geometry with Voigt's limb-darkening formula (7), we obtain for a specified window function

$$\mathrm{w}(i,\varepsilon,\varphi,\delta) = \begin{cases} 0.4(1+1.5\cos\vartheta) & \text{for} \quad \cos\vartheta \geq 0 \\ 0 & \text{for} \quad \cos\vartheta < 0 \end{cases} . \tag{9}$$

Although the direction of the line of sight determines the observability of the stellar magnetic field, we can not separate this part of the window function from the calculation of the field. So, it takes up an intermediate place connecting them.

6.4. Matrix formulation of the window function

Fig. 3 shows the view at the globe with the elementary squares included by the coordinate lines, which can be arranged in form of a matrix as formulated in equation (1). In contrary to the vector matrix $\boldsymbol{B}_{\delta\varphi}$, however, the matrix elements $\mathrm{W}_{\delta\varphi}$ are scalar magnitudes with the meaning of transmission factors,

$$\mathbf{W}_{\delta\varphi} = \begin{pmatrix} \mathrm{W}_{11} & \mathrm{W}_{12} & \dots & \mathrm{W}_{1\varphi} \\ \mathrm{W}_{21} & \mathrm{W}_{22} & \dots & \mathrm{W}_{2\varphi} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{W}_{\delta 1} & \mathrm{W}_{\delta 2} & \dots & \mathrm{W}_{\delta\varphi} \end{pmatrix} \quad . \tag{10}$$

The representation of the window as a matrix is obviously a commendable advantage for the computerized numerical calculation.

But there is, moreover, still another advantage: For changing aspect of the star during its rotation under the inclination angle i we need to calculate the window matrix $\mathbf{W}_{\delta\varphi}(i)$ only once. The coordinates according to equation (9) are then related to the rows and columns of a constant window matrix (10), under which the cartographic map of the star slides, varying only the longitude φ - the columns - step by step.

That is a *matrix-convolution*, which represents the temporally varying aspect of the star during its rotation by *discreetized convolution.*

7. The vectorial character of the magnetic field

The window, through which we see the star, is strictly directed by the vector of the spatially fixed line of sight. Thus, the magnetic field vector, penetrating the star's atmosphere, can have all possible directions in respect to the line of sight, changing them periodically by rotation of the star. The projection of the vector field onto the line of sight is a constituent part of the window function.

We observe from the Earth the longitudinal component of the magnetic field vector by the three spherical components B_r, B_φ, and B_δ in polar coordinates fixed to the stellar globe

$$\boldsymbol{B}_{\text{sight}} = B_r \mathbf{a}_r + B_\varphi \mathbf{a}_\varphi + B_\delta \mathbf{a}_\delta \quad . \tag{11}$$

The unity vectors $\mathbf{a}_r$, $\mathbf{a}_\varphi$, and $\mathbf{a}_\delta$ of the polar coordinate system are transformed in Cartesian coordinates by the rectangular unity vectors **i**, **j, k**

$$\mathbf{a}_r = \cos\delta\cos\varphi\mathbf{i} + \cos\delta\sin\varphi\mathbf{j} + \sin\delta\mathbf{k} \tag{12}$$

$$\mathbf{a}_\varphi = -\cos\delta\sin\varphi\mathbf{i} + \cos\delta\cos\varphi\mathbf{j} \tag{13}$$

$$\mathbf{a}_\delta = -\sin\delta\cos\varphi\mathbf{i} - \sin\delta\sin\varphi\mathbf{j} + \cos\delta\mathbf{k} \quad . \tag{14}$$

The projection of the magnetic field vector related to each point of the surface is carried out by a scalar multiplication of the magnetic field vector $\boldsymbol{B}$ with its components B_a, B_φ, and B_δ adjusted to the vector of the line of sight,

$$\mathbf{v} = \sin i \cos t\mathbf{i} + \sin t \sin t\mathbf{j} - \cos i\mathbf{k}, \tag{15}$$

$$\boldsymbol{B}_{\text{eff}} = \boldsymbol{B}\ \mathbf{v} = B_r[\cos\delta\sin i(\cos\varphi\cos t + \sin\varphi\sin t) - \sin\delta\cos i] +$$
$$+\ B_\varphi[\cos\delta\sin i(\cos\varphi\sin t - \sin\varphi\cos t)] +$$
$$+\ B_\delta[-\sin\delta\sin i(\cos\varphi\cos t + \sin\varphi\sin t) - \cos\delta\cos i]\ , \qquad (16)$$

which is the "effective magnetic field strength" $\boldsymbol{B}_{\text{eff}}$ as measured already by Babcock using the Zeeman displacement of the spectral line profiles between the left-handed and right-handed circularly polarized light (Stokes V). In the same way, by scalar products of the magnetic field vector with the two perpendicular directions to the line of sight, also the Stokes components Q and U of the linearly polarized light are obtained (Gerth & Glagolevskij 2003, 2004).

7.1. Magnetic field components and other stellar magnitudes

We continue here only with the magnetic field strength as the Stokes component V and denote this function by B - in place of the other vectorial components of the magnetic field.

The components of the magnetic vector B_r, B_δ, B_φ will be calculated in a computer program separately. They may be used for further calculation, combined among them to derive the *surface field intensity*

$$B_{\text{S}} = \sqrt{B_r^2 + B_\delta^2 + B_\varphi^2} \qquad (17)$$

or the *surface horizontal field*

$$B_{\text{h}} = \sqrt{B_\delta^2 + B_\varphi^2}\ , \qquad (18)$$

but also done in connection with other stellar magnitudes and conditions.

Deviating from other methods but favorable for the computer is the calculation of the four components I Q U V of the so-called *"Stokes-vector",* which is performed by projection of the total magnetic field vector at the surface of the star onto the vector of the line of sight. This is done on the base of the rules of vector algebra by scalar products of the corresponding vectors (Gerth & Glagolevskij, 2000).

The magnetic vector map of $\boldsymbol{B}(\delta, \varphi)$ can be exploited still otherwise. Very important is the derivation of the *geometric line profile* by "sorting in" of magnetic magnitudes – vector components as well as derived ones – into a frequency distribution (see chapter 9.3).

7.2. Combination of the magnetic vector with scalar magnitudes

Besides of the three components of the magnetic field vector, a fourth component is foreseen for such magnitudes as brightness, temperature or others, which are distributed over the surface of the star as a cartographic map. To this category there belongs also the distribution of chemical elements, being the main point of the present paper in chapter 10.3.

The scalar magnitude, however, is not only an auxiliary magnitude for the magnetic field, for it may be used independently. If the scalar is an inhomogeneous brightness distribution with dark and/or bright spots on the surface, then the phase curve of the integral luminosity is calculated.

7.3. The velocity vector field on the stellar surface

Quite another magnitude, but acting in the same manner on the line profile with displacement and deformation like the magnetic field, is the velocity of moving material on the surface, caused by rotation or/and meridional and convectional flows. The profile of a spectral line will be displaced by the *Doppler*-effect and disfigured by the geometrical conditions. It is seen through the aspect window similarly as in the case of the magnetic surface field. Thus, the same algorithms can be used for computation. The difference is only the arrangement of the vectorial field.

The main velocity field of a star is caused by its rotation. This is a toroidal field, which varies the equatorial velocity v_{equ} from the equator to the poles by the factor $\cos\delta$. For a homogeneous radiation of the surface, we get the *rotational line profile*, which looks like a half ellipse. Because of the geometrical conditions, the profile has its maximal extension by viewing directly equator-on and disappears looking pole-on. Inhomogeneities on the surface like spots, convection cells, and even eclipses transiting over the visible disk, disfigure the profile with the well-known characteristics.

The algorithms of the program are suited also for the calculation of the comprised profile variations of double stars. Then, each of the companions will have its own velocity vector field with the overlaid orbital velocity.

The velocity fields should not be laid aside in the investigation of magnetic fields at stars, because they act simultaneously and produce overlaid effects.

7.4. Graphical representation on cartographic maps

On the basis of a program developed by the authors, all magnitudes of the magnetic field with their derivations and combinations with other magnitudes can be demonstrated graphically. This is very important for controlling and comparing of the results obtained during the computing procedure, but it gives, moreover, also the possibility of archiving and producing figures for publication.

8. The representation of stellar rotation by convolution

8.1. The convolution integral

If we translate the idea of a convolution to the integral representation of the effective magnetic field strength after Oetken (1977) analogously to equation (4), but with the rotation axis erected perpendicularly to the Cartesian z-coordinate and a line of sight inclined by the angle i, then we have

$$\boldsymbol{B}_{\mathrm{int}}(t) = \frac{\int\limits_{\delta=-\pi/2}^{\pi/2} \int\limits_{\varphi=0}^{2\pi} \boldsymbol{B}(\delta,\varphi)\mathrm{w}(i,\varepsilon,\delta,\varphi-t)\mathrm{d}\varphi\mathrm{d}\delta}{\int\limits_{\delta=-\pi/2}^{\pi/2} \int\limits_{\varphi=0}^{2\pi} \mathrm{w}(i,\varepsilon,\delta,\varphi-t)\mathrm{d}\varphi\mathrm{d}\delta}\,. \tag{19}$$

This integral formula gives the integrated mean of the disk seen by the observer and comprises the *convolution integral*, representing the rotation of the star with its map $\boldsymbol{B}(\delta,\varphi)$ behind the window $\mathrm{w}(i,\varepsilon,\delta,\varphi)$ after equation (9).

Likewise as in equation (4), the evaluated denominator is a constant factor, which normalizes the integral.

The rotation of the star yields an equator-parallel shifting of the aspect, showing a variable longitude φ with progressing time t, which is expressed by the characteristic difference (φ - t) of a convolution integral in the argument of the window function $\mathrm{w}(i, \varepsilon, \delta, \varphi - t)$.

Equation (19) is more instructive than equation (4), but it lacks likewise by the direct integration of the magnetic field components. As we see later and as it has been showed already in (Gerth & Glagolevskij, 2003c), some kind of a "geometrical line profile" is included hiddenly in both formulae (4) and (19).

8.2. Matrix-convolution

Equation (19) can be integrated analytically, if $\boldsymbol{B}$ is an integrable function such as spherical harmonics (Legendre functions). This, however, is restricted to the surface of the sphere and continuous structures with analytical formulation. The general numerical integration can be performed using standard algorithms. Then the computer discreetizes the functions and applies interpolating algorithms like Simpson's rule, which are suited properly for time-economizing computation.

Anyway, discreetization is unavoidable for digitalization in every case. Therefore, let's make a virtue of necessity - using right from the beginning numerable magnitudes such as matrices are! We prefer the matrix calculation for programming because of its very clear coordination of the field structure to the cartographic map. In case of low ranks, the economy in computing time is even better for algorithms of matrix procedures than for those of integrals. The computing time, however, increases quadratically with the rank. Thus, in the state of initial investigations with low matrix ranks, iteration procedures run faster, yielding preliminary results, which might be refined later using higher ranks.

At first, we have to define another kind of multiplication of two rectangular matrices as known usually: The elements of both matrices are multiplied individually corresponding to the elements of the factors by coinciding indices.

This is necessary for the matrix formulation of the observation window covering the cartographic map, whereby every element B_{ik} of the map matrix coincides with the corresponding element W_{ik} of the window matrix, represented by the product $B_{\mathrm{ik}}\mathrm{W}_{\mathrm{ik}}$.

Therefore, using the matrix elements of equations (1) and (10), we formulate the combined matrix of map and window in the following way:

$$(\boldsymbol{B}\mathbf{W})_{\delta\varphi} = \begin{pmatrix} B_{11}\mathrm{W}_{11} & B_{12}\mathrm{W}_{12} & \dots & B_{1\varphi}\mathrm{W}_{1\varphi} \\ B_{21}\mathrm{W}_{21} & B_{22}\mathrm{W}_{22} & \dots & B_{2\varphi}\mathrm{W}_{2\varphi} \\ \vdots & \vdots & \ddots & \vdots \\ B_{\delta 1}\mathrm{W}_{\delta 1} & B_{\delta 2}\mathrm{W}_{\delta 2} & \dots & B_{\delta\varphi}\mathrm{W}_{\delta\varphi} \end{pmatrix} . \tag{20}$$

When the window matrix is shifted by t columns in direction of the longitude φ, then the combined matrix is

$$(\boldsymbol{B}\mathbf{W})_{\delta(\varphi-t)} = \begin{pmatrix} B_{11}\mathrm{W}_{1(1-t)} & B_{12}\mathrm{W}_{1(2-t)} & \dots & B_{1\varphi}\mathrm{W}_{1(\varphi-t)} \\ B_{21}\mathrm{W}_{2(1-t)} & B_{22}\mathrm{W}_{2(2-t)} & \dots & B_{2\varphi}\mathrm{W}_{2(\varphi-t)} \\ \vdots & \vdots & \ddots & \vdots \\ B_{\delta 1}\mathrm{W}_{\delta(1-t)} & B_{\delta 2}\mathrm{W}_{\delta(2-t)} & \dots & B_{\delta\varphi}\mathrm{W}_{\delta(\varphi-t)} \end{pmatrix} . \tag{21}$$

Keeping in mind, that the trapezoid square elements of the globe become narrower starting from equator and going on to the poles, then the sum of one column is given by left-hand multiplication of the $\boldsymbol{B}\mathbf{W}$-matrix with the row matrix $\boldsymbol{s}$ equation (3).
Summation of the squares along the longitude rings parallel to the equator is made with planes of equal magnitude, so that all elements of the column matrix are 1 (unity),

$$(\mathbf{1})_{\varphi} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} . \tag{22}$$

Multiplication of the matrix (21) on the left side with the row matrix (3) and on the right side with the unity column matrix (20) yields by linking of the matrices the sum

$$\boldsymbol{B}_{\mathrm{sum}}(t) = (\boldsymbol{s})(\boldsymbol{B}\mathbf{W})(\mathbf{1}) = \sum_{\varphi=1}^{2n} \sum_{\delta=1}^{n} s_{\delta} B_{\delta\varphi} \mathrm{W}_{\delta(\varphi-t)} , \tag{23}$$

which is a function of time, namely the phase curve of the integral magnetic field strength $\boldsymbol{B}_{\mathrm{eff}}(t)$ for the three components of the magnetic vector B_r, B_{φ}, and B_{δ}. With increasing rank $n \to \infty$ the matrix formulation equation (23) goes over to the convolution integral equation (19) – neglecting the denominator as a constant factor. Avoiding by this way ambiguity and confusion, we use for the matrix representation of the integral transformation only the numerator corresponding to equation (19).

In principle, there is no difference between the factors $\boldsymbol{B}$ and $\mathbf{W}$ concerning the shift by t, because the convolution integral is commutative. We prefer here the version that the window slides over the map of the globe. Thus, the rectangular *Mercator map* has a fixed position, and the phase curve can be coordinated and graphically drawn on the map, which then is concurrent with the phase diagram.

9. The integral spectral line profiles

All information about the magnetic field of a star is contained in the Zeeman-displaced line profiles originating from its atmosphere. The composition of the line profile, however, is very complicated. We list here only the main influences.

For full comprehension there has to be taken into account:

1. the atomic processes of absorption and emission,
2. the radiation transfer through the stellar atmosphere
3. the geometric conditions -
   a. the projection at the sphere to the line of sight,
   b. the topographic arrangement of the observable field,
   c. the stratification of the atmospheric layers, -
4. the integration over the star's visible disk.

9.1. The elementary line profile

Every point in the atmosphere of the globe of a star has its own characteristic of line production, which depends on the rate of the line-generating chemical elements and the radiation transfer conditions according to points 1. and 2. of the foregoing chapter. We will take this line profile - independently of its special formation - as the *elementary line profile,* not asking, what the forming physical processes are. The form of the profile might even degenerate in the very extreme case to a "needle"-like one.

Whatsoever, the really observed line profile of the whole disk of a star is a mixture of profiles outgoing from all visible surface points, which deforms the original elementary profiles somehow. This effect is caused by integration and averaging over all radiating surface areas of the star, mainly in consequence of the geometrical conditions (point 3. in chapter 8). By this way even needle-like profiles become broader. In former publications (of ours and other authors) the influence of the profile was neglected at all. Practically, there was used only the *delta*-function - unknowingly. Thus, the meaning came up that there the magnetic field itself was integrated.

However, we have to realize finally: **The integration of the observable stellar magnetic field takes place only by means of the spectral line profile.**

9.2. A model function for line profiles

Since the computation of line profiles on the physical basis of atomistic processes and radiative transfer is very complicated because it demands a vast amount of parameters with special conditions, we propose here a very simple approximation formula for an *elementary line profile*, which needs only *one* parameter n:

$$f(x) = \frac{1}{(1+x^2)^n} \ . \tag{24}$$

Melcher and Gerth (1977) showed that this formula proves to be a remarkably good fitting function for most of the really observed symmetrical line profiles, if all possible values of n are admitted, even non-integer numbers.

The philosophy of this formula is as simple as its formulation. It is a generalized *Lorentz*-distribution, which follows from the *Fourier*-transformation of a *Poisson*-function of the degree n. For $n \to \infty$ the *Poisson*-function goes over to the *Gauss*-function. Roughly physically, it represents a multi-step reaction by excitation of atoms, causing the absorption of light of determined frequency. The number n of steps is different for the quantity of atoms reacting concurrently. Thus, statistics render also non-integer values of n.

The independent variable $x = \alpha\triangle\lambda$ in equation (24) has the meaning of a wavelength difference $\triangle\lambda$ multiplied by the half-value width α of the profile.
Anyway, we can use this very handy formula successfully in our computer programs for *model calculation of elementary line profiles.*

9.3. The geometrically caused line profile

The geometrical conditions of projection and visibility listed in chapter 8. have an enormous influence upon the profile of the integral magnetic field strength. The magnetic field vector on the surface is strictly bound to the geometry of the star. The information-bearing light emerging from all parts of the surface in direction to the observer is - so to speak - "filtered and gathered".

If we take the component of the field projected on the line of sight B_{sight} from every point of the surface, then we have a function of the magnetic field strength with a two-dimensional domain of definition

$$B_{sight} = f(\delta, \varphi) \; . \tag{25}$$

The values of B_{sight} from all individual points are distributed over a certain area of the B-scale, making a frequency distribution. It is correct to take this distribution as a $profile$, because the magnetic field is aligned with the wavelength of the spectral lines by the Zeeman-displacement and/or line-broadening.
We will show here a simplified derivation. So we take a scale of $B = B_{sight}$, on which the profile-functions $w(b)$ from all surface elements coordinated to δ,φ of equation (23) are added up by their places on the abscissa of the variable B,

$$w(b) = \sum_{\delta,\varphi} B_{\delta,\varphi} w_{\delta,\varphi}(b - B_{\delta,\varphi}) \; . \tag{26}$$

The "sorting in" of the values in a distribution is like the accidental flight of doves into the holes of a pigeonry and needs no order. The sum-formula (24) relates only to *all* squares of (δ, φ) equally but without regular arrangement.

Going over from the sum to an integral, we have to account for the one-dimensional distribution of B, which implements the two-dimensional definition domain – like a "scanned" plane with equal weight for all differential parts. Then, we define an elementary profile function $w(b)$ uniformly for the whole surface of the star and replace the sum by an integral – getting the *convolution integral*

$$w(b) = \int_{\varphi=0,\delta=-\frac{\pi}{2}}^{\varphi=2\pi,\delta=+\frac{\pi}{2}} B(\delta, \varphi) w_{\delta,\varphi}(B(\delta, \varphi) - b) \mathrm{d}B \; . \tag{27}$$

This "geometric line profile" depends on the aspect of the star and is generally asymmetric – as it was shown by Gerth and Glagolevskij (2000, 2004). Therefore, the line profile is periodically changing during the rotation of the star. The geometry of the star with its aspect disfigures every "elementary line profile". Even the "needle"-like *delta*-function is broadened and distorted, producing thus a "clean" *geometric profile* without any other profile-generating processes.

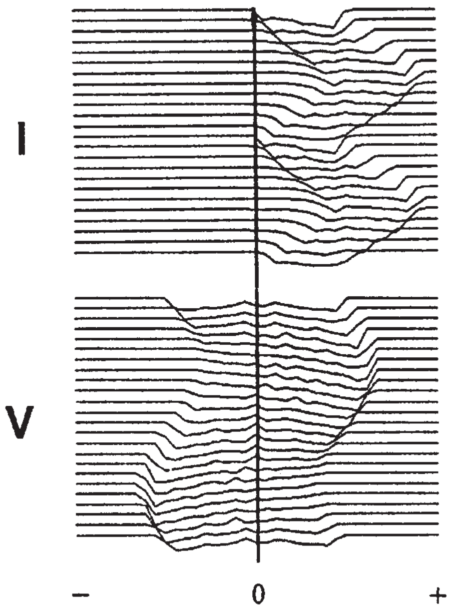

Figure 4. Group of "geometric line profiles" of a magnetic dipole in the course of a period for Stokes I *and* Stokes V. *The profiles reproduce broad phase curves like a winding riverbed. Poles:* + ($\varphi = 90^o$ $\delta = 45^o$), – ($\varphi = 270^o$ $\delta = -45^o$)

10. Convolution of line profiles

In the integral radiation emerging from the star, the line profiles of all surface elements are mixed anyhow to an *integral line profile*, which is described by equation (27) as a *convolution integral.*
In order to explain the principal relation more convincingly, let us first consider a one-dimensional definition function $F(x)$ instead. Then the mixing with the profile $f(x)$ takes place by the convolution integral

$$f(y) = \int_{-\infty}^{+\infty} f(x)F(x-y)\mathrm{d}y \; . \tag{28}$$

This is a definite *integral-transformation* of the function $f(x)$ with the kernel $F(x-y)$ for the independent variable x with an infinite domain of definition. By restriction to an finite domain and discreetization, this definite integral can be expressed by a left-hand matrix multiplication of a row matrix $\boldsymbol{f}_x$ with the diagonal matrix $\boldsymbol{F}_{xy}$ yielding the row matrix $\boldsymbol{g}_y$:

$$\begin{pmatrix} g_1 & g_2 & \dots & g_y \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & \dots & f_x \end{pmatrix} \begin{pmatrix} F_{11} & 0 & \dots & 0 \\ 0 & F_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F_{xy} \end{pmatrix} \; . \tag{29}$$

The expansion of the elements in the row represent the discreetized *profile function*, which can be taken as grating of classes (so-called "pigeon-holes") of a *frequency distribution*.
The sum formula of the matrix convolution is then given by

$$(\boldsymbol{f}_{y-t}) = (\boldsymbol{f}_x)(\boldsymbol{F}_{x(y-t)}) = \sum_{x=1}^{n} f_x F_{x(y-t)} \; , \tag{30}$$

which transforms the x-row matrix to the y-row matrix by shifting the diagonal elements of the square matrix in equation (29) step by step across the principal diagonal.
This matrix multiplication is used by ours for the computer algorithm.

10.1. The convolution integral, combining rotation and line profile

Now we have to take into consideration, that the definition domain for the line profile distribution-function is the two-dimensional sphere. The line profile convolution is added to the rotational convolution integral equation (19) by a further integration, so that there are three integral signs. Both convolutions act on the here considered *simulated model star* simultaneously.

The distribution of the polarized radiation over a region b around $\boldsymbol{B}$ is defined by the elementary profile function $\omega(b)$ coordinated to the spectral wavelength of the line λ and convoluted with the window function $\mathrm{w}(i, \varepsilon, \delta, \varphi)$ in the phase integral equation (19):

$$\boldsymbol{B}_{\mathrm{int}}(t, \lambda) = \frac{\int\limits_{\delta=-\pi/2}^{\pi/2} \int\limits_{\varphi=0}^{2\pi} \int\limits_{-\infty}^{+\infty} \boldsymbol{B}(\delta, \varphi, \lambda) \mathrm{w}(i, \varepsilon, \delta, \varphi - t) \omega(b - \lambda) \mathrm{d}\lambda \mathrm{d}\varphi \mathrm{d}\delta}{\int\limits_{\delta=-\pi/2}^{\pi/2} \int\limits_{\varphi=0}^{2\pi} \int\limits_{-\infty}^{+\infty} \mathrm{w}(i, \varepsilon, \delta, \varphi - t) \omega(b - \lambda) \mathrm{d}\lambda \mathrm{d}\varphi \mathrm{d}\delta} \tag{31}$$

This equation is the analytical representation of the phase curve of the integral magnetic field strength as well for the whole magnetic vector $\boldsymbol{B}$ as for the vectorial components B_r, B_δ, B_φ, including the *two convolutions* due both to rotation and to the frequency distribution of the field strength, being effective for the integral radiation viewed by the observer in the line of sight.

10.2. Accounting for the distribution of chemical elements

The integral equation (31) is valid for a homogeneous radiation emerging from the surface with an equal distribution of chemical elements and surface temperature. This is fulfilled rather completely for a stellar atmosphere of hydrogen. Therefore, magnetic field phase curves derived from Zeeman-measurements of hydrogen lines show a very smooth sinusoidal-like appearance - in contrast to those of metallic lines.

This is because the metallic elements of *chemically peculiar stars* (CP stars) are spread inhomogenously over the surface. The reason for this phenomenon is not ensured yet. But we can assume, that there takes place a ring-like concentration around the magnetic poles by interaction of the magnetic and electric properties of atoms and molecules with the magnetic field on the way of diffusion and accretion processes.

Thus, it is evident, that we can expect to observe only line profiles from such chemical elements, which are present on a determined site of the surface. Besides of the information about the magnetic field strength by the magneto-relevant Zeeman-displacement, we have to account for the deepness of the line, which stands for the concentration of chemical elements. All these data are comprised in the convolution integral.

10.3. The fourth component of the cartographic field map

The coordination of the distribution of chemical elements to the cartographic localization on the globe of the star is 1 to 1. This means, that every site on the globe is coordinated to the magnetic vector with *three* components and to *one* scalar magnitude - the concentration of chemical elements. So we have ***four*** *magnitudes*, which allow us to determine the *effective magnetic field strength* $\boldsymbol{B}_{\rm eff}$.

Likewise to the magnetic surface field, the concentration of chemical elements $T(r, \delta, \varphi)$ is represented as a cartographic map itself. The dependence of the concentration on the radius r accounts even for the stratification in the atmosphere layer. For our purposes, however, the two variables δ and φ suffice.

The scalar function $T(r, \delta, \varphi)$ can be treated as a *transmission factor* for the radiation transfer through the atmosphere layer. This factor may incorporate also other things, which hinder the *transfer of radiation*, for instance spots, clouds and eclipses. Therefore, we call this scalar magnitude "***transmission*** *factor*".

The "factor map" occupies the fourth component of the *field matrix*, which is arranged in the computer program with $2n^2{+}1$ rows and 4 columns as a fourth column after the three columns of the magnetic field components.

10.4. Matrix-representation of the twofold convolution

Accounting for the distribution of elements on the surface, the magnetic map $\boldsymbol{B}(r, \delta, \varphi)$ in the double-convolution integral equation (28) will be replaced by the unambiguously coordinated product $\boldsymbol{B}T(r, \delta, \varphi)$.

Analogously to the map of the surface magnetic field equation (1), also the map of the distribution of chemical elements – or more general: the *transmission factor* – has a two-dimensional domain of definition, which can be represented best by a rectangular matrix

$$\boldsymbol{T}_{\delta\varphi} = \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1\varphi} \\ T_{21} & T_{22} & \dots & T_{2\varphi} \\ \vdots & \vdots & \ddots & \vdots \\ T_{\delta 1} & T_{\delta 2} & \dots & T_{\delta\varphi} \end{pmatrix} , \qquad (32)$$

which is – otherwise as the window matrix $\mathbf{W}$ – rigidly connected with the matrix of the map $\boldsymbol{B}$ by multiplication of every coordinated element of each matrix.

The coupling of the one-dimensional line profile to the two-dimensional map requires a "three-dimensional matrix". This would be a scheme of elements with three indices for the distribution $\mathbf{D}_{\delta\varphi\lambda}$, which links the matrices of different rank and degree. $\mathbf{D}_{\delta\varphi\lambda}$ is actually a *tensor of third degree.* [3]

Corresponding to the definite convolution integral (31) we can choose for this tensor an arbitrary rank m. Practically, it is recommended to take the same rank n as for the map matrix, because then in the computer the same storage areas and algorithms can be used – as we did. Then, this tensor provides the linking also to the window matrix and replaces the unity matrix equation (20)

$$(\boldsymbol{d})_\lambda = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_\lambda \end{pmatrix} . \tag{33}$$

Thus, completing equation (23) by the transmission matrix $\boldsymbol{T}$ and the distribution tensor $\mathbf{D}$, we obtain the sum formula of the *twofold convolution integral formula including the distribution of chemical elements*

$$\boldsymbol{B}_{\text{sum}}(t) = (\boldsymbol{s})(\boldsymbol{BT}\mathbf{WD})(\boldsymbol{d}) = \sum_{\lambda=1}^{m}\sum_{\varphi=1}^{2n}\sum_{\delta=1}^{n} s_\delta B_{\delta\varphi}\, T_{\delta\varphi} \mathrm{W}_{\delta(\varphi-t)} \mathrm{D}_{\delta(\varphi-t)\lambda} d_\lambda \ , \tag{34}$$

which is the *very general formula* underlying the algorithms of our computer program.

This formula, moreover, holds and is still something more general, if we take not only a unique elementary profile for the whole surface of the star. It is possible, to coordinate the profiles to the map and/or to the window, so that the "pigeon-hole"-effect is implemented in the matrix-transformation using the matrices $\mathbf{W}$ and $\boldsymbol{T}$ with individual matrix elements. The convolution of the elementary profile is already completed before the insertion in the integral and does not appear explicitly in the sum-formula equation (34).

The coordination of the profile to individual sites on the surface of the star could be important for such cases, that the radiative transfer through the atmosphere effects non-negligibly on the profile form – as for instance at the limb of the disk, where the optical path through the atmosphere layer is prolonged.

11. Inversion problems

From the beginning of investigating magnetic stars it was the desire to reconstruct the magnetic field of a star by inverse calculation. Therefore, the compiled and in phase diagrams arranged observational results should be subjected to an inversion procedure.

The methods for the inversion were first developed for *Doppler Imaging* and later extended to the *magnetic field structure* by the pioneers in this field,

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[3] In this nomenclature a square matrix is a tensor of second degree, and a vector, a row or a column matrix would be a tensor of first degree.

namely V.L. Khokhlova and her followers N.E. Piskunov and O.P. Kochukhov. We relate here only to some papers of these authors (contributed on the *Conference on Magnetic Stars* in Nizhnij Arkhyz (2000)).

On the base of the approach to the reconstruction of the stellar magnetic field as outlined in this paper, it will be interesting, whether this is compatible with that of the other authors.

11.1. Inversion of the rotational convolution

In the course of the straightforward calculation we derived the convolution integral formulae for the phase curve of the magnetic field strength – equations (19) and (31) – as integral transforms. Generalizing and reducing these equations to the substantial basis connected with the rotation of the star, and neglecting in these equations the denominator, which makes only the normalization, then we can write them in a simplified form as definite integral depending on the only changing variable φ in the course of t

$$\boldsymbol{B}(t) = \int\limits_{\varphi=0}^{2\pi} \mathrm{w}(i,\delta,\varphi-t)\boldsymbol{B}(\delta,\varphi)\mathrm{d}\varphi \,, \tag{35}$$

where $\mathrm{w}(i,\delta,\varphi-t)$ is the kernel of a *Fredholm integral equation of first type.* Translating this integral equation to the matrix form, we have

$$B_{\delta t} = \sum_{\varphi} \mathrm{W}_{\delta(\varphi-t)} B_{\delta\varphi} \,. \tag{36}$$

This is the linear transformation from the row matrix B_φ into the row matrix B_t by means of the matrix kernel as the window matrix $\mathrm{W}_{\delta(\varphi-t)}$, which includes the geometrical conditions of aspect by the inclination angle i. The longitude δ, however, goes through the transformation without any effect. By this way a set $B_{\delta\varphi}$ of δ column matrices, which establish a rectangular matrix itself, can be transformed simultaneously.

If we write equation (35) by matrix multiplication

$$\boldsymbol{B}_t = \mathbf{W}\boldsymbol{B} \,, \tag{37}$$

then the inversion is given by

$$\boldsymbol{B} = \mathbf{W}^{-1}\boldsymbol{B}_t \;\; . \tag{38}$$

This is a the matrix representation of a linear system of differential equations, which is to be solved by the reciprocal window matrix $\mathbf{W}^{-1}$. Since $\mathbf{W}^{-1}$ consists only of a-priori-known variables, the recalculation of the field distribution along the longitude φ can be executed by a computer using the algorithm of matrix inversion. By this way, for instance, the longitudinal position of the poles is determined. The latitude δ, however, stays indifferent.

11.2. Inversion of the geometrically caused convolution

As we see, the inversion of the rotational convolution integral is not sufficient for the determination of both coordinates of the magnetic map: δ and φ. Therefore, both δ and φ should vary. This is achieved, if we consider also the geometrically caused convolution, the information of which is contained in the line profile. The area of the visible disk in the line of sight is seen through the *window* – but not more. Other parts of the surface become visible by rotation of the star, for the window function related to the line of sight with the angels ϑ (to the line) and ϕ (around the line) covers only one hemisphere of the globe, tilted by i to the rotational axis.

If we reduce equation (31) to the profile domain λ, then the star is viewed at a determined phase of rotation. It seems that the information contained in the visible disk is not sufficient for inversion. However, the window in direction i to the observer supplies the independent equations required for the inverse solution.
The window function with all its coordinate transforms and geometrical projections determines the kernel of a definite two-dimensional integral equation of Fredholm's first type

$$\boldsymbol{B}(\lambda) = \int\limits_{\delta,\varphi} \int\limits_{\vartheta,\phi} \mathrm{w}(i, \vartheta, \phi - \lambda) \boldsymbol{B}(\delta, \varphi, \lambda) \mathrm{d}\lambda \ , \tag{39}$$

where the integrals with their limits δ, φ and ϑ, ϕ are spread over the entire surface of the sphere.
The integral equation (39) can now be written in a matrix formulation like equation (37) and inverted by the reciprocal kernel matrix like equation (38). –

At this point, however, we stop our consideration of about inversion of the geometrical convolution. This problem has been tackled successfully by Piskunov and Kochukhov (2000), whom we refer to.

11.3. Critical remarks to the inversion of integral magnitudes

The inversion of integral magnitudes like the *effective magnetic field strength* in order to reconstruct the magnetic field on the surface of a star raises some principal problems.

1. The source material of the observation should be accurate and confident, because every inversion amplifies scatter and noise, pretending thus spurious features.

2. Since the inversion of an integral equation corresponds to the solution of a system of differential equations with n independent variables, a set of n independent equations is required, which must not fall short. Using matrices for the solution, then n is the rank of the resolvent kernel matrix.

3. The set of independent equations can be achieved by changing the aspect window of the star for the visible hemisphere. Repetitive observations are not independent and improve only the statistical accuracy.

4. For a star tilted by the inclination angle i a part of the surface remains unknown for ever.

5. Magnetic field strength and element concentration on the surface of the star are bound together. The structure established by the inversion treatment of the data can only give the product of both. Independent derivation like Doppler-Imaging is commendable to separate the magnetic field from the inhomogeneous element distribution. The reconstruction of the complete magnetic field vector with three components requires the inversion of the source data of the four Stokes components in polarized light. Despite of all progress of measuring techniques in this field, the inversion of the entire magnetic field vector is still complicated and rather uncertain.

6. Provided, the inversion has been performed successfully, then only the surface distribution of the magnetic field is revealed, which allows us to describe the magnetic field structure of a special observed star in form of a cartographic map.

7. The continuation of the magnetic field towards the interior and to the exterior of the star would be an uncertain extrapolation which does not uncover the origin of the field.

Nevertheless, the inversion of the observed data in order to determine the surface structure of the stellar magnetic field is a very valuable method for the investigation of stellar magnetism and should be applied and extended to many stars to seek for common and typical phenomena.

Inversion and straightforward computation seem to be opposite poles. However, they belong together as a *dialectic couple.* The uncertain result of the inversion should be tested and proved by the always certain straightforward calculation, which could be at hand like a control tool.

12. Something about the computer program

The development of the program started in autumn 1994 in cooperation of Yu. V. Glagolevskij and E. Gerth. In that time the inversion method of V.L. Khokhlova was already published. After the first idea to control the inversion results by an own program, it was the intention of the authors from the beginning of this task to *model the magnetic field structure* and to derive the phase curve of the integral magnetic field strength in a straightforward calculation, restricting themselves only to this item.

The main point of the program was the construction of the magnetic field out of its sources, which are assumed to be virtual “magnetic charges” – in analogy the electric charges of an electric field. A standard algorithm was set up for the field of an elementary point-like source. By superposition of the fields of numerous sources all magnetic field structures can be modeled, shaping thus the magnetic field on the surface, but also inside and outside the star.

The program has been developed by “doing and testing”, using for controlling graphical representations and proving all partial mathematical procedures and the corresponding algorithms individually.

Most important there proved to be the graphical representation of the map on the monitor screen during the development of the completing program for testing and improving. By this way the program grew step by step. Sometimes the mathematical formulation was set up only after testing and improving of the algorithms, guaranteeing thus a high degree of reliability.

The program was written in the language GWBASIC and run in interpreter mode, but then it has been compiled for practical use.

13. Conclusion

Concluding we summarize the main points of the calculation of the integral magnetic field strength of an inhomogeneously with chemical elements covered star by the following theses:

1. The observed magnetic field is an integrated one, called the integral magnetic field $\boldsymbol{B}_{\mathrm{int}}$, which exists independently of visibility and detection.
2. The integration is not related to the magnetic field itself but to the information transferring medium: the spectral line profile.
3. The element distribution acts like a transparency filter for the field. Accounting for integration, projection, polarization, and measuring effects, we measure the effective magnetic field $\boldsymbol{B}_{\mathrm{eff}}$.
4. The phase curve of the integral magnetic field strength $\boldsymbol{B}_{\mathrm{int}}(t)$ is the result of the convolution due to the rotation of the star behind the aspect window.
5. The deformation of the spectral line profile is caused by a convolution with the geometrical conditions of the aspect to the globe of the star.
6. For the practical computation, the cartographic map of the magnetic surface field and the convolution integrals are discreetized and arranged in rectangular matrices.
7. A computer program for the straightforward calculation of magnetic surface fields on stars and the phase curves of the integral magnetic field strength is a *valuable tool* for investigation of magnetic stars by fitting derivations from hypothetical parameters to real observational results.

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